• Solution of the Schrödinger equation for the hydrogen atom leads to the same energy quantization as in the Bohr model, but with total angular momentum quantized according to

$$L \;=\; \hbar\sqrt{l(l+1)} \quad , \quad l = 0,1,2,3,\ldots \tag{32}$$

and the azimuthal component of L quantized according to $L_z = m_l\hbar$ for $|m_l| = 0,1,\ldots,l$. There is a degeneracy of energy levels with l and m_l. Note also that $l \leq n-1$ bounds the angular momentum quantum number.

• Introducing an external magnetic field $\mathbf{B}$ defines a preferred direction, breaking the isotropy symmetry and thereby splitting the energy degeneracy. The fine structure of lines is described by the splitting

$$\Delta E \;\sim\; \pm\frac{eB\hbar}{2\mu c} \quad , \tag{33}$$

which applies to the atomic electrons. This effect is called the **Zeeman effect** and is used to measure solar and stellar magnetic fields.

$\boxed{\textbf{Plot:}}$ Zeeman Splitting of Atomic Lines

• Where does this energy splitting estimate come from? A semi-classical treatment suffices to answer this. Energy differences can be obtained by computing the work done by moving a charge a distance $\Delta\mathbf{r}$ under the Lorentz force:

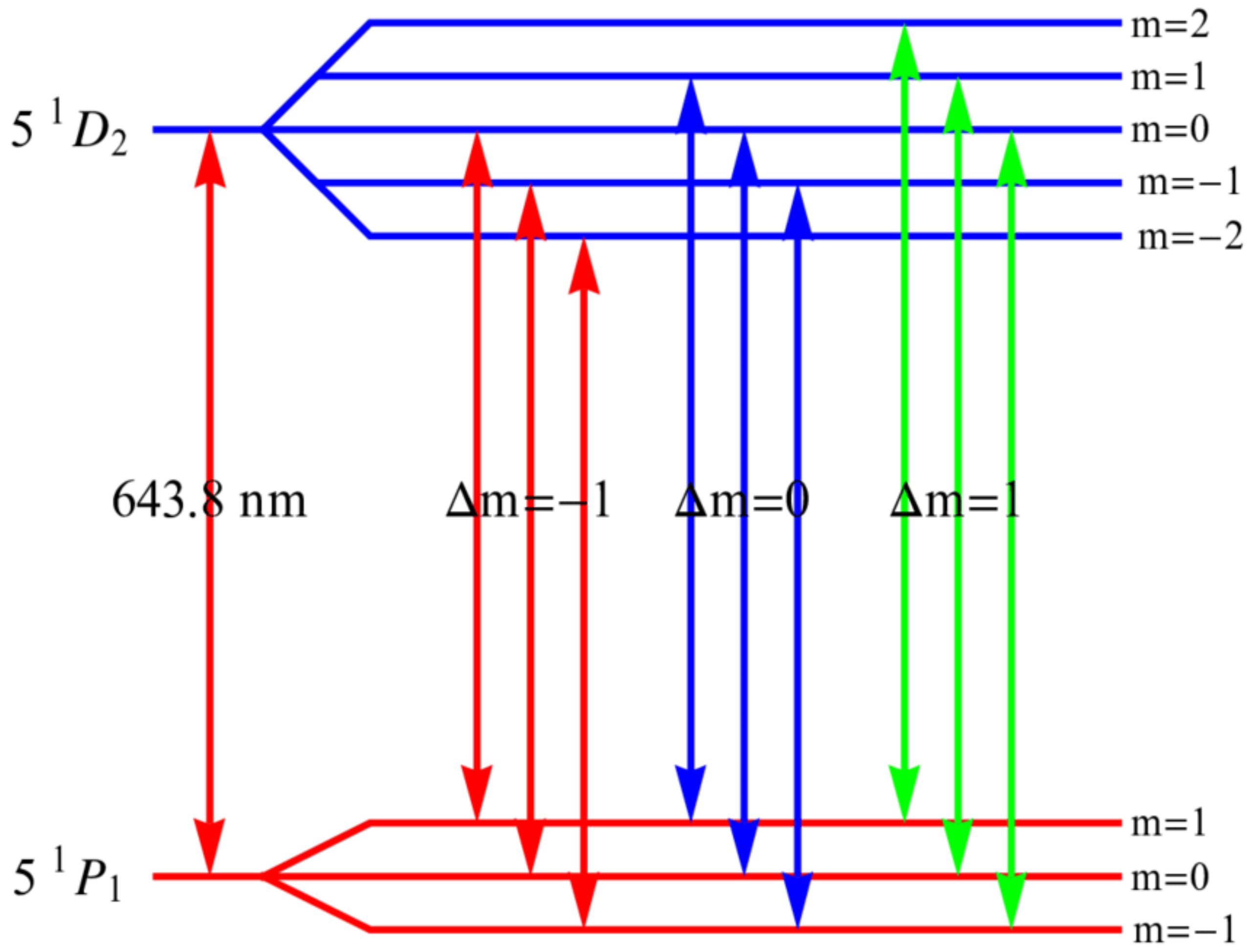
$$|\Delta E| \;=\; \Delta\mathbf{r}\cdot\frac{d\mathbf{p}}{dt} \;=\; \frac{e}{c}\,\Delta\mathbf{r}\cdot\Bigl(\mathbf{v}\times\mathbf{B}\Bigr) \;\to\; eB\,|\Delta\mathbf{r}|\,\frac{|\mathbf{v}|}{c} \quad . \tag{34}$$

The Bohr model of the atom can be used to estimate the kinetic energy of the electron through $\mu v^2/2 \approx |E_{\rm tot}| = \mu e^4/(2\hbar^2)$. Thus, one estimates the orbital speed of a Schrödinger electron to be $v/c \sim e^2/(\hbar c) = \alpha_{\rm f} \approx 7.3\times 10^{-3}$, *which is non-relativistic*. The displacement of a bound orbital electron is on the scale of the Bohr radius $a_0 = \hbar^2/(\mu e^2)$. Accordingly,

$$|\Delta E| \;\to\; eB\,|\Delta\mathbf{r}|\,\frac{|\mathbf{v}|}{c} \;\sim\; eB\,a_0\,\alpha_{\rm f} \;=\; \frac{eB\hbar}{\mu c} \quad . \tag{35}$$

This derivation clearly connects to the physical elements of the Bohr atomic model. Yet we note that a quick way to derive this estimate is just to multiply the cyclotron frequency $eB/\mu c$ by $\hbar$.

Zeeman Effect: Line Splitting by *B*



- Splitting of atomic lines by magnetic fields was first observed by P. Zeeman (1897) in Cadmium, shown here. *Credit*: I. Suzuki

6. STELLAR SPECTRA AND ATMOSPHERES

Matthew Baring – Lecture Notes for ASTR 350, Fall 2025

1 Spectral Classification

Spectral classification started as an organized taxonomy with the work at Harvard of Pickering — stars were labelled by letters according to the strength of their H absorption lines (AFGKM) **C & O, Sec. 8.1**

• Annie Jump Cannon (1901) revised this classification by sequencing them according to temperature:

$$OBAFGKM \tag{1}$$

∗ O stars: hot, blue, young and massive (labelled *early-type*)

∗ M stars: cool, red, old and less massive (labelled *late-type*)

Subdivisions are numbered: e.g. $B1 \to B9$.

Temperature strongly influences the state of atoms, whether they are ionized or not, hence we expect a strong coupling between spectral type and line characteristics. States critically depend on species, so line spectra give powerful indicators of "real temperatures" as opposed to effective temperature.

• Ionization states are classified observationally via Roman numerals:

∗ H I = neutral hydrogen; H II = ionized hydrogen

∗ He I = neutral helium; He II = singly-ionized helium; He III = doubly-ionized helium. e.g. Si IV, OVII, MgII

2 Atom Excitation: the Boltzmann Equation

Spectral classification depends on (i) in what orbitals are electrons most likely to reside? and (ii) what are the relative states of ionization?

• Answers are governed by statistical mechanics of thermal gases, which indicate that the velocity distribution of a gas of non-relativistic particles of mass m at temperature T is given by

$$\boxed{n_v dv \; = \; 4\pi\, n \left(\frac{m}{2\pi kT} \right)^{3/2} \; \exp\left\{ -\frac{mv^2}{2kT} \right\} v^2\, dv \quad ,} \tag{2}$$

where k is Boltzmann's constant. This is the famous **Maxwell-Boltzmann** distribution of statistical/thermal physics.

$\boxed{\textbf{Plot:}}$ Maxwell-Boltzmann Distribution

The peak velocity is $v_{\rm peak} = \sqrt{2kT/m}$ and the rms value is $v_{\rm rms} = \sqrt{3kT/m}$.

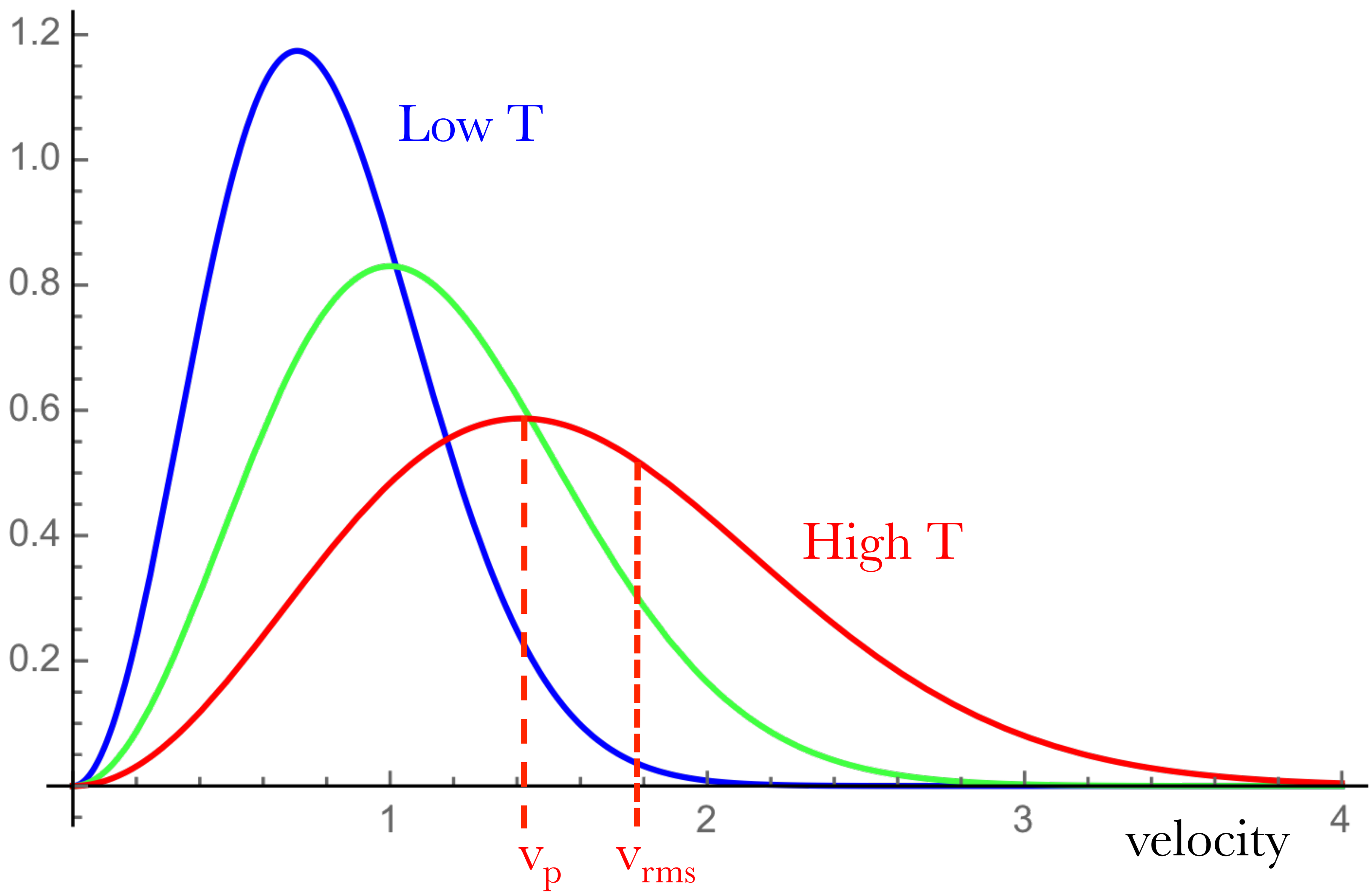
$$v_{\rm rms}^2 \; = \; \langle v^2 \rangle \; = \; \int_0^\infty v^2\, n_v\, dv \bigg/ \int_0^\infty n_v\, dv \; = \; \frac{3kT}{m} \quad . \tag{3}$$

It then follows that the mean kinetic energy is $\langle K \rangle = mv_{\rm rms}^2/2 = 3kT/2$, which is the ideal gas equation of state.

• Thermonuclear reaction rates in stellar interiors critically depend on such distributions, in detailed balance similar to the atomic considerations below.

• In the atomic context, v represents atom speeds in a hot or cool gas, which then impacts the distribution of electrons in atomic states via a collisional excitation/de-excitation. In this way, the exponential in the M-B distribution maps over to an $\exp(-E/kT)$ factor that can apply to both the atom kinetic energies and also the orbital excitation energies.

Maxwell-Boltzmann Distributions



Let E_a and E_b be two energy levels of an atom, each with g_a and g_b degenerate sub-states; i.e. $g_{a,b} \geq 1$. If $P(E_a)$ and $P(E_b)$ are the probabilities of finding e^- in these respective energy levels, then statistical mechanics yields

C & O, pp. 209–12

$$\boxed{\frac{N_b}{N_a} \equiv \frac{P(E_b)}{P(E_a)} = \frac{g_b}{g_a} \exp\left\{-\frac{E_b - E_a}{kT}\right\} \quad .} \tag{4}$$

This is called the **Boltzmann equation**. The factor $e^{-E/kT}$ is called the **Boltzmann factor**. Note that $g_{a,b} \neq 1$ arises through spin and L and L_z degeneracies in the solution of the Schrödinger equation.

Ground States s_1				Energy E_1
n	ℓ	m_ℓ	m_s	(eV)
1	0	0	$+1/2$	-13.6
1	0	0	$-1/2$	-13.6
First Excited States s_2				**Energy E_2**
n	ℓ	m_ℓ	m_s	(eV)
2	0	0	$+1/2$	-3.40
2	0	0	$-1/2$	-3.40
2	1	1	$+1/2$	-3.40
2	1	1	$-1/2$	-3.40
2	1	0	$+1/2$	-3.40
2	1	0	$-1/2$	-3.40
2	1	-1	$+1/2$	-3.40
2	1	-1	$-1/2$	-3.40

Quantum Number Degeneracies for Hydrogen

• e.g. Consider the hydrogen atom and excitations from the ground state $n = 1$, with $g_1 = 2(1)^2 = 2$ to the first excited state $n = 2$ with $g_2 = 2(2)^2 = 8$. Here $E_1 = -13.6\,\text{eV}$ and $E_2 = -13.6\,\text{eV}/4 = -3.4\,\text{eV}$.

At what temperature does $N_2 = N_1$?

$$1 \;=\; \frac{N_2}{N_1} \;=\; \frac{8}{2}\,\exp\left\{ -\frac{(-3.4\text{eV}) - (-13.6\text{eV})}{kT} \right\} \quad , \tag{5}$$

yielding $T = 8.54 \times 10^4\,\text{K}$. This is clearly hotter than effective temperature of sunlight, implying that most hydrogen atoms at the solar surface are in the ground state.

$*$ Yet hydrogen Balmer lines achieve maximum intensity at much lower temperatures, around $10^4\,\text{K}$. Something else must be in play!

$\boxed{\textbf{Plot:}}$ Boltzmann Equation for Hydrogen

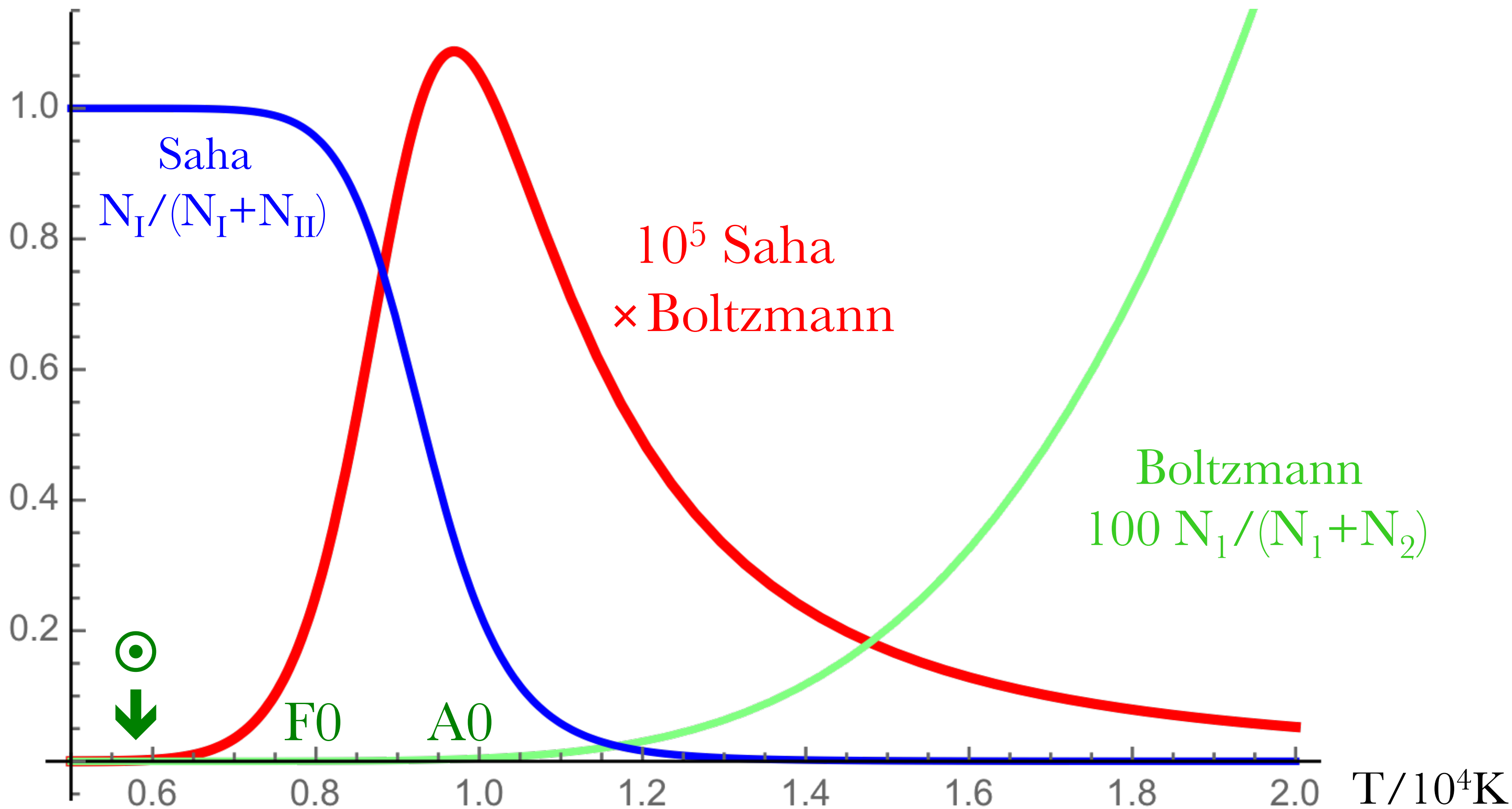
• Now consider singly-ionized helium, which will require a significant temperature to realize such a state. The Schrödinger equation applies to it, and the solution scales like the hydrogen atom. Yet, now the nuclear charge is Ze for $Z = 2$, and $\mu \approx m_e$. The Bohr model tells us that $E_1 = -54.4\,\text{eV}$ and $E_2 = -54.4\,\text{eV}/4 = -13.6\,\text{eV}$.

At what temperature does $N_2 = N_1$ now?

$$1 \;=\; \frac{N_2}{N_1} \;=\; \frac{8}{2}\,\exp\left\{ -\frac{(-13.6\text{eV}) - (-54.4\text{eV})}{kT} \right\} \quad , \tag{6}$$

yielding $T = 3.4 \times 10^5\,\text{K}$, i.e. 4 times larger than for the hydrogen example. This case pertains to white dwarf stars, and again is much hotter than their typical surface temperatures.

Boltzmann and Saha Equations



- Boltzmann excitation (green, 1-2) and Saha (neutral fraction, blue, I-II) ionization solutions for hydrogen for temperatures T in units of 10^4K. Here $n_e=10^{14}\text{cm}^{-3}$.
- Combined (red) illustrates peak Balmer H_α line signal at $T\sim10^4$K.

3 Ionization and the Saha Equation

Ionization balance also critically affects the strengths of atomic de-excitation lines, constraining the number of states available for de-excitation. Such ionization balance also depends strongly on Boltzmann factors. **C & O, pp. 213–6**

Let χ_i be the ionization energy for transitioning from ionization state i to state $i+1$. Thus, for $HI \to HII$, $\chi_I = 13.6\,\mathrm{eV}$.

• We define the **partition function** Z_i to be the weighted sum of the number of ways an atom or ion ionization state i can arrange its electrons among excitation states. Boltzmann statistics then gives

$$Z_i \;=\; g_1 + \sum_{j=2}^{\infty} g_j \, e^{-(E_{i,j}-E_{i,1})/kT} \tag{7}$$

for the g_i being the degeneracy factors ($g_1 = 2$, $g_2 = 8$, etc.; $g_n = 2n^2$).

• The ratio of the number N_{i+1} of atoms in ionization state $i+1$ to the number N_i in state i is given by detailed balance and is

$$\boxed{\frac{N_{i+1}}{N_i} \;=\; \frac{2Z_{i+1}}{n_e Z_i} \left(\frac{m_e kT}{2\pi\hbar^2}\right)^{3/2} e^{-\chi_i/kT} \quad .} \tag{8}$$

This is the **Saha Equation**, derived in 1920. The factor of 2 accommodates the two spin states of free electrons, and represents their partition function (i.e. no bound states).