11. COMPACT OBJECTS I

Matthew Baring – Lecture Notes for ASTR 350, Fall 2021

1 White Dwarfs

Bessel (1844) concluded after an extensive campaign of observation of Sirius, that it's proper motion was not in a straight line, but had a sinusoidal wobble. He inferred that Sirius was a binary system, with a period of 50 years and that the companion was faint; it was called the pup to the *Dog Star.* **C & O, Sec. 16.1**

∗ In 1862, Alvan Clark discovered the companion and found it to be 1000 times fainter than Sirius A; it was named Sirius B, and had a dynamical mass of around $M_{\odot}$. Clark's observations were made at apastron, when visual binaries are their most detectable.

∗ Near the next apastron, in 1915, Adams was able to perform spectroscopy on Sirius B. Contrary to expectations that it would be a cool, red star, it was found to be a hot, blue star.

• We now know that Sirius B has an effective temperature of $27,000\,\mathrm{K}$ in the UV. The Stefan-Boltzmann law yields a radius of $5.5 \times 10^8\mathrm{cm} \approx 0.008 R_{\odot}$, smaller than that of the Earth. Hence, $\rho_{\mathrm{WD}} \sim 10^6 \rho_{\odot} \sim 10^6\ \mathrm{g\ cm}^{-3}$.

Plot: White Dwarfs on the H-R Diagram

• Such a **compact object** is known as a **white dwarf**. Its immense gravity broadens hydrogen absorption lines dramatically.

∗ They are typically underluminous and emit predominantly in the UV, which explains why they were detected only in the 20th Century.

Sirius B: First Known White Dwarf

• Conditions in the interior can be estimated as follows. The hydrostatic pressure required to support the star against gravity is (for $M \sim M_\odot$)

$$P_c \approx \frac{2\pi}{3} \frac{GM^2}{R^4} \approx 3.8 \times 10^{23} \text{ dyne cm}^{-2} \sim 10^8 P_{c,\odot} \quad . \tag{1}$$

Using the ideal gas equation, one arrives at a central temperature of $T = P/(Nk) = (P/k) \times 4\pi R^3 m_H/(3M\mu) \sim 10^8\,\text{K}$ for $M \sim M_\odot$.

∗ In reality, their central temperatures are a factor of a few smaller; *i.e. thermal gas pressure cannot support the star against the pull of gravity.*

• Such temperatures would lead to prolific pp chain activity if hydrogen dominated the white dwarf core. The underluminous nature of these stars suggests that hydrogen fusion cannot be proceeding *in their cores.*

∗ This implies that their centers must consist of largely C and O.

Plot: Global Properties of Compact Stars

1.1 Degenerate Electrons in White Dwarfs

White dwarfs are supported by **electron degeneracy pressure**. As the temperature becomes comparatively small, the proportion of available quantum states that are filled increases. When the kinetic energy drops to low values, then the gas no longer remains classical, but follows a **Fermi-Dirac** distribution:

C & O, Sec. 16.3

$$n(\varepsilon) = \frac{1}{\exp\{(\varepsilon - \varepsilon_F)/kT\} + 1} \tag{2}$$

Plot: Fermi-Dirac Distribution

• The density of states d^3p in momentum space is specified by quantum statistics. At absolute degeneracy, since $\lambda = 2\pi\hbar/p$ is the de Broglie wavelength, then $d^3p/(2\pi\hbar)^3$ is a number of states per unit volume:

$$n = \frac{2}{(2\pi\hbar)^3} \int_0^{p_F} 4\pi p^2\, dp \quad \Rightarrow \quad p_F = \hbar \left(3\pi^2 n\right)^{1/3} \quad , \tag{3}$$

for the **Fermi momentum** p_F of particles with 2 spin states.

Table 1: Global Properties of Compact Stars

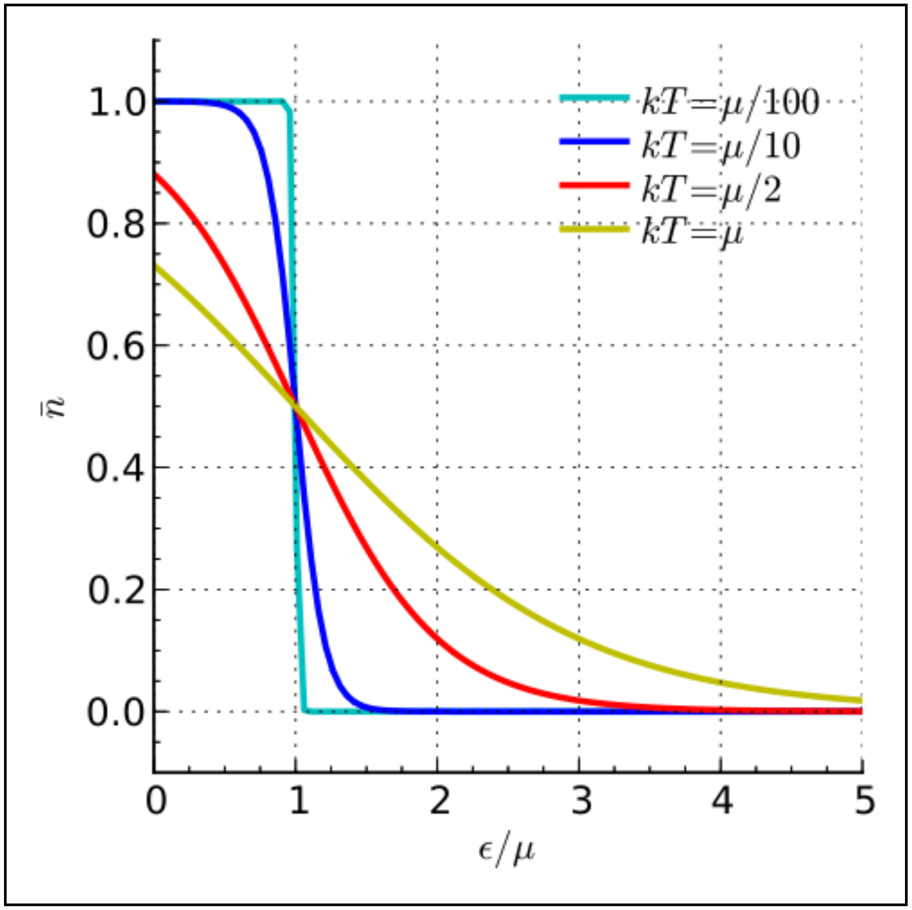
Object	Mass[a] (M)	Radius[b] (R)	Mean density (g cm^{-3})	Surface Potential (GM/Rc^2)
Sun	$M_\odot$	$R_\odot$	1	10^{-6}
White Dwarf	$\lesssim 1.4 M_\odot$	$\sim 10^{-2} R_\odot$	$\sim 10^6$	$\sim 10^{-4}$
Neutron star	$\sim 1 - 3 M_\odot$	$\sim 10^5 R_\odot$	$\sim 10^{15}$	~ 0.1
Black hole	arbitrary	$2GM/c^2$	$\sim M/R^3$	~ 1

Notes: [a] $M_\odot \approx 1.989 \times 10^{33}$ g. [b] $R_\odot \approx 6.960 \times 10^{10}$ cm.

Fermi-Dirac Distribution Function

$$n_{\mathrm{FD}}(E,T) = \frac{1}{\exp[(E-\mu)/kT]+1}$$

- The Fermi-Dirac distribution for the occupation number *n* as a function of the fermion energy ε, scaled by the chemical potential μ (from Kittel, *Introduction to Solid State Physics* [4th ed., 1971].)
- As kT -> 0, the distribution approaches the degenerate step function form, with *n* non-zero only when ε<μ.
- At high T, distribution approaches the Maxwell-Boltzmann form.



• Since the Pauli exclusion principle prohibits electrons being in the same state, as the thermal momentum drops to zero, a "repulsive" force sets in ... this is a fermionic *degeneracy pressure.*

• At $T = 0\,\mathrm{K}$, all the ground states are occupied and none of the excited states are, and the gas is said to be **completely degenerate**. The distribution becomes a step function, being zero above the **Fermi energy**:

$$\boxed{\varepsilon_F \;=\; \frac{p_F^2}{2m} \;=\; \frac{\hbar^2}{2m}\left(3\pi^2 n\right)^{2/3} \quad .} \tag{4}$$

Here m and n are the mass and number density of the fermions. For electrons, $n_e = Z\rho/(Am_H)$ for complete ionization.

• The effective criterion for degeneracy is $kT \lesssim \varepsilon_F$. Setting $n = n_e$ in Eq. (4), this translates to

$$\frac{T}{\rho^{2/3}} \;\lesssim\; \frac{\hbar^2}{3m_e k}\left[\frac{3\pi^2}{m_H}\,\frac{Z}{A}\right]^{2/3} \;=\; 1.3\times 10^5\ \mathrm{K\ cm^2\ g^{-2/3}} \quad . \tag{5}$$

For the central temperature and radius obtained above for Sirius B, namely $T \sim 10^8\,\mathrm{K}$ and $\rho \sim 10^6\,\mathrm{g\ cm^{-3}}$, it can be deduced to be degenerate.

• Carroll & Ostlie use a heuristic argument to estimate the degeneracy pressure P. However, it can be simply obtained. Thermodynamics establishes $P = -\partial U/\partial V$ for energy U <u>per particle</u> and volume $V = 1/n$ <u>per particle</u>. At zero temperature, for a non-relativistic gas

$$U \;=\; \frac{\langle p^2\rangle}{2m} \;=\; \frac{1}{2m}\int_0^{p_F} p^2\, d^3p \Big/ \int_0^{p_F} d^3p \;=\; \frac{3p_F^2}{10m} \;=\; \frac{3\varepsilon_F}{5} \quad . \tag{6}$$

Hence, the degenerate, non-relativistic equation of state (EOS) is

$$P \;=\; \frac{3}{5}\, n^2\, \frac{\partial \varepsilon_F}{\partial n} \;=\; \frac{(3\pi^2)^{2/3}}{5}\,\frac{\hbar^2}{m_e}\, n_e^{5/3} \quad . \tag{7}$$

For Sirius B, this computes (for $Z/A = 0.5$ for massive nuclei) to 2×10^{23} dyne cm^{-2}, i.e. approximately that required for hydrostatic equilibrium.

• *Clearly, electron degeneracy pressure is the prime candidate for support of white dwarfs against gravitational collapse.*

1.2 White Dwarf Mass-Radius Relation

Considering hydrostatic equilibrium, one can equate the gravitational and degeneracy pressures: **C & O, Sec. 16.4**

$$P \sim \frac{\hbar^2}{m_e}\left(\frac{\rho}{m_p}\right)^{5/3} \sim \frac{\hbar^2}{m_e}\left(\frac{M}{R^3 m_p}\,\frac{Z}{A}\right)^{5/3} \sim \frac{GM^2}{R^4} \quad . \tag{8}$$

Inverting this leads to a *relationship between the mass and the radius*:

$$\boxed{M_{\rm wd}\, R_{\rm wd}^3 \;\propto\; \frac{1}{m_p^5}\left(\frac{\hbar^2}{Gm_e}\right)^3 \;\approx\; 0.45 M_\odot \left(10^9 {\rm cm}\right)^3 \quad ,} \tag{9}$$

assuming that most of the star is degenerate, i.e. that CO or other nuclear burning does not ensue.

• Counter-intuitive character: *white dwarfs of greater mass are smaller*! Increased gravity requires higher pressure → greater density → smaller radius.

Plot: Mass-Radius Relation

• Density concentration at the center alters the scaling coefficient in Eq. (9) and so a precise solution for a polytrope is needed.

• The mass-radius relation does depend on the mean molecular weight $\mu \to A/Z$, and therefore the white dwarf composition. This induces a natural spread in the Hertzsprung-Russell diagram that leads to some uncertainty in SN Ia being precise standard candles for cosmology.

Plot: White Dwarf Masses and Radii from Gaia Observations

White Dwarf Mass-Radius Relation: *Hipparcos* Update

- Provencal et al. (ApJ **494**, 759, 1998)

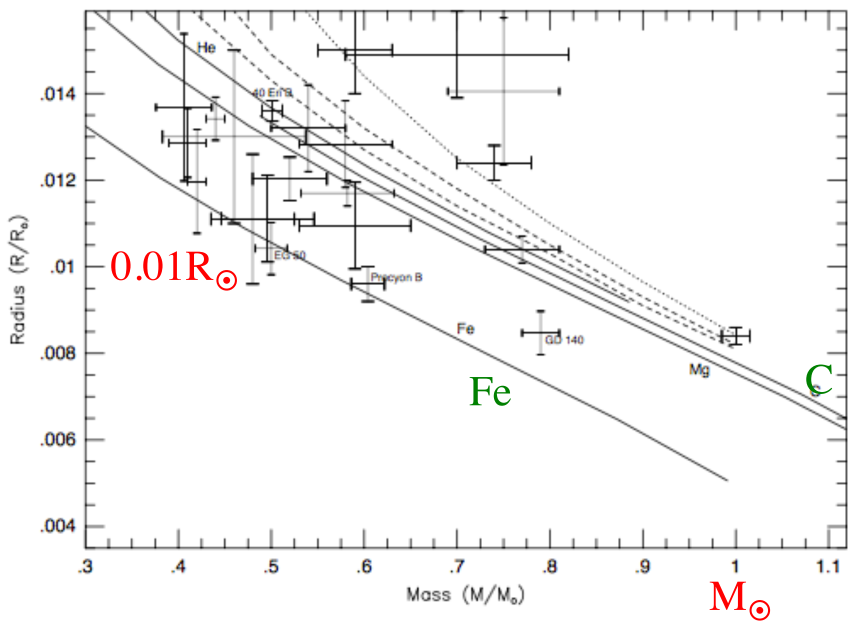


FIG. 3.—Observational support for the white dwarf mass-radius relation, showing the positions of the visual binaries, common proper-motion systems, and field white dwarfs. The field white dwarf masses were derived using published surface gravity measurements and radii based on *Hipparcos* parallaxes.

TABLE 7

COMPARISON OF WHITE DWARF MASSES

Object	M_{log} ($M_\odot$)	M_{spec} ($M_\odot$)	M_{gr} ($M_\odot$)	M_{astro} ($M_\odot$)
Binary Systems				
Procyon B				0.604 ± 0.018
Sirius B..............				1.000 ± 0.016
40 Eri B	0.47 ± 0.01	0.51 ± 0.03	0.50 ± 0.01	0.501 ± 0.011
CD−38 10980......	0.71 ± 0.02	0.66 ± 0.02	0.74 ± 0.04	
G181-B5B...........	0.28 ± 0.10	0.47 ± 0.03	0.50 ± 0.03	
G154-B5B...........	0.31 ± 0.10	0.43 ± 0.03	0.46 ± 0.09	
G156-64	1.19 ± 0.18	0.86 ± 0.04	0.59 ± 0.06	
L268-92			0.70 ± 0.12	
Wolf 485 A	0.69 ± 0.02	0.54 ± 0.03	0.59 ± 0.04	
Field Stars				
GD279	0.44 ± 0.02	0.53 ± 0.03		
Feige 22	0.41 ± 0.03	0.48 ± 0.03		
EG 50	0.50 ± 0.02	0.66 ± 0.03		
EG 21	0.58 ± 0.05	0.63 ± 0.03		
GD 140	0.79 ± 0.02	0.90 ± 0.03		
G238-44	0.42 ± 0.01	0.55 ± 0.03		
G226-29	0.75 ± 0.03	0.70 ± 0.03		
WD2007-303........	0.44 ± 0.05	0.51 ± 0.03		
Wolf 1346	0.44 ± 0.01	0.51 ± 0.03		
G93-48	0.75 ± 0.06	0.61 ± 0.03		
L711-10	0.54 ± 0.04	0.56 ± 0.03		

NOTE.—M_{spec} refers to spectroscopic masses from BSL and Bragaglia et al. 1995. M_{log} refers to masses derived using published surface gravities and radii calculated from *Hipparcos* parallaxes. M_{gr} is gravitational mass, and M_{astro} is the astrometric mass.

White Dwarf Mass-Radius Relation: *Gaia* Update

- Tremblay et al. (MNRAS **465**, 2849, 2017). ESA astrometry probe Gaia improves parallax determination by over an order of magnitude (Data release 1).
- This permits refined distance and luminosity determinations => more accurate radius measurements relative to Hipparcos (1990s).
- Spectroscopy and binary dynamics give M/R so that mass estimates improve too.
- Solid circles are directly-observed WDs, open circles are WDs in wide binaries. Colors match those of model curves.
- Gaia, launched in December 2013, will collect data on around 500,000 WDs and a billion MS stars. DR 2 in April 2018.