3 Stellar Energy Sources

There are several possible sources of energy that could fuel the radiative power of the sun: rotational energy (angular momentum), gravitational potential energy, magnetic or electrical energy, chemical energy or nuclear energy. Generally, rotational, electromagnetic and chemical energy can be quickly ruled out as being insufficient, though we note that rotational energy powers pulsars, and magnetic energy can power magnetars.

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3.1 Gravitational Potential

• Consider *gravitational energy.* For mass element dM within a star located at radius r from the center, the contribution to the potential energy is

$$dU_g = -\frac{G\,M(r)dM}{r} \quad , \tag{17}$$

where $M(r)$ is the enclosed mass satisfying Eq. (6). Also, $dM = 4\pi r^2 \rho\, dr$, so that we can integrate to yield

$$U_g = -4\pi G \int_0^R M(r)\rho(r)\, r\, dr \quad . \tag{18}$$

If we approximate the star as one of uniform density, then its mass is $M = 4\pi\overline{\rho}R^3/3$ and also $M(r) = 4\pi\overline{\rho}r^3/3$. Inserted into Eq. (18), these yield

$$U_g = -\frac{3}{5}\frac{GM^2}{R} \quad , \tag{19}$$

a result that will be modified by the density distribution within the star.

∗ $|U_g|$ increases for mass concentration near the center, the case for collapsing pre-supernova cores.

∗ Given that the kinetic energies of elements of the stellar interior satisfy the virial theorem, the available energy is the negative of the total mechanical energy, i.e. $\Delta E_g = 3GM^2/(10R)$.

• For the sun this value is 1.1×10^{48} ergs, which incidentally is considerably less than the total energy liberated in a supernova. Assuming a constant solar luminosity throughout its life, we quickly obtain a gravitational *lifetime* for the sun:

$$t_{\rm KH} \; = \; \frac{\Delta E_g}{L_\odot} \; \sim \; 9.1 \times 10^6 \text{years} \quad . \tag{20}$$

This is know as the **Kelvin-Helmholtz** timescale. Since it is much shorter than known geological lower bounds to the solar age, we conclude that the sun *is not powered by gravity.*

3.2 Nuclear Timescales

Consider now *nuclear energy.* Nuclear constituents are protons and neutrons. Hydrogen is most abundant in the universe. If neutrons are to form in the nucleus, the nuclear proton must absorb an electron in a nuclear interaction (involving production of neutrinos). The essential reaction is

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$$p + e \; \to \; n \tag{21}$$

with a mass balance differential

$$\begin{aligned} \Delta m \; &= \; m_p + m_e - m_n \\ &= \; 1.672623 \times 10^{-24} g + 9.109390 \times 10^{-28} g - 1.674929 \times 10^{-24} g \\ &= \; -1.3951 \times 10^{-27} g \quad . \end{aligned} \tag{22}$$

This clearly suggests that it is not energetically favorable to produce neutrons (*i.e., free neutrons should decay*)!

• However, this does not take into account the *binding energy* of nucleons in close proximity. We can assess this by comparing the helium mass with that of hydrogen.

• First define the *atomic mass unit*: $1\text{u} = 1.660540 \times 10^{-24}$ g as one twelfth of the mass of isotopic carbon 12. Note that $1 \text{ u} = 931.49432\,\text{MeV}/c^2$.

• In these units, we have $m_H = 1.007825\,\mathrm{u}$ and $m_{He} = 4.002603\,\mathrm{u}$, less than $4m_H = 4.031280\,\mathrm{u}$. Hence

$$\Delta m \;=\; 4m_H - m_{He} \;=\; 0.028677\,\mathrm{u} \quad\Rightarrow\quad \frac{\Delta m}{4m_H} \approx\; 0.007 \quad . \tag{23}$$

It is concluded that the binding energy of helium is around 26.71 MeV, or almost 7 MeV per nucleon. *This far exceeds atomic binding energies.*

• We can now calculate a nuclear burning lifetime for the sun assuming this efficiency of 0.7%. Assuming a sun composed entirely of hydrogen, and that 10% of this can undergo nuclear fusion to helium, we have

$$E_{\rm nucl} \;=\; 0.1 \times 0.007 \times M_\odot c^2 \;=\; 1.3 \times 10^{51}\ \mathrm{ergs} \quad . \tag{24}$$

so that the **nuclear timescale** is

$$t_{\rm nucl} \;=\; \frac{E_{\rm nucl}}{L_\odot} \;\sim\; 10^{10}\ \mathrm{years} \quad . \tag{25}$$

This is long enough to account for geological ages for the solar system.

3.3 Thermonuclear Reaction Rates

Now we turn our attention to the probabilities of nuclear reactions. Since nuclei are positively-charged, there is a strong Coulomb repulsive force that inhibits reactions. This must be overcome before an even stronger nuclear attractive force dominates on short length scales, namely 10^{-13} cm.

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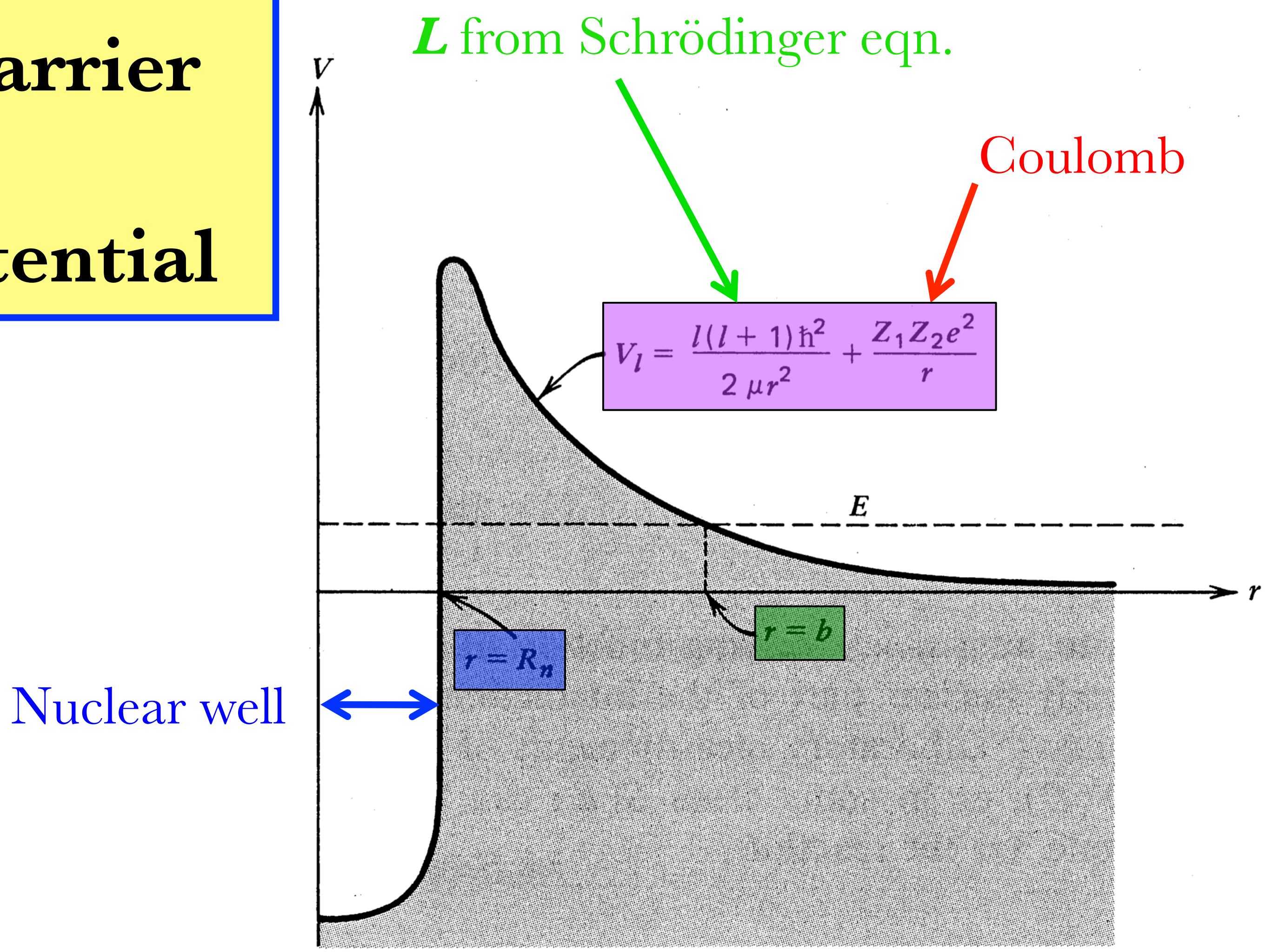
Plot: Nuclear Potential Energy Profile

A rough <u>*classical*</u> estimate of a sufficient temperature in a stellar interior to surmount the Coulomb barrier can be found by equating the kinetic energy to the Coulomb potential: for $\mu \sim m_p/2$ as the reduced mass,

$$\frac{1}{2}\,\mu v^2 \;\sim\; \frac{3}{2}\,kT_{\rm cl} \;\sim\; \frac{Z_1 Z_2 e^2}{r} \;\equiv\; U_c \quad , \tag{26}$$

where the scale r is the nuclear radius of 10^{-13} cm. This then establishes that $T \sim 10^{10}$ K is needed, much higher than that of the solar center.

Coulomb Barrier and Nuclear Potential



- The effective potential V(r) governing the radial motion of one nucleus relative to another (Fig. 3.4 of Shapiro & Teukolsky). The short range attractive nuclear force dominates at $r<R_n$. At larger distances, the potential is dominated by the repulsive Coulomb force.
- The classical turning point for an orbit is at $r = b = Z_1Z_2e^2/E$ for a CM energy E.

• This classical calculation doesn't account for *quantum tunnelling* through the Coulomb barrier. Heisenberg's uncertainty principle yields $\Delta x \Delta p_x \gtrsim \hbar/2$, so that penetration on scales Δx can be achieved by particles of momenta $\hbar/\Delta x$. Nucleons are not confined to points in space.

• This coupling is manifested in the de Broglie wavelength $\lambda = h/p$, which can be ported into the energy equation just above:

$$\frac{p^2}{2\mu} \;=\; \frac{h^2}{2\mu\lambda^2} \;\sim\; \frac{3}{2}\, kT_q \;\sim\; \frac{Z_1 Z_2 e^2}{\lambda} \quad . \tag{27}$$

This can be solved for λ to yield $1/\lambda \sim \mu Z_1 Z_2 e^2/(2\pi^2\hbar^2)$, and therefore

$$kT_q \;=\; \frac{\mu Z_1^2 Z_2^2 e^4}{3\pi^2\,\hbar^2} \;\approx\; \frac{2 Z_1^2 Z_2^2}{3\pi^2}\,\frac{\mu}{m_e}\,\chi_H \quad , \quad \chi_H \;=\; 13.6\,\mathrm{eV}, \tag{28}$$

i.e., a value of $T_q \sim 10^7\,$K. This value can easily explain nuclear burning in the solar interior. Here μ is the reduced mass ($\approx m_p/2$ for pp collisions).

• The fact that the classical and quantum values are so disparate emphasizes the fact that the nuclear reactions are inherently quantum, and moreover, should be very sensitive to T:

∗ The temperatures in stellar interiors may sample the exponential tails of Maxwell-Boltzmann distributions n_v of nuclear speeds.

• To determine the behavior of the rates, we note that the cross section $\sigma(v)$ must be integrated over all v, with the probability of interaction being proportional to v. If n_v is the MB distribution of target nuclei, and n is the number density of "projectile" nuclei, then the rate of reaction is

$$\frac{dn}{dt} \;=\; \int_0^\infty n\, n_v\, \sigma(v)\, v\, dv \quad . \tag{29}$$

The most influential factor in this integral is the cross section. Crudely taking it to be the effective target area of interaction, we might expect

$$\sigma(v) \;\propto\; \pi\lambda^2 \;\propto\; \left(\frac{h}{p}\right)^2 \;\propto\; \frac{1}{E} \tag{30}$$

in the nonrelativistic limit. However this is too small ($\sim 10^{-26}\,\mathrm{cm}^2$) and ignores tunnelling. Quantum tunnelling turns out to depend exponentially on the Coulomb barrier height U_c and the projectile energy $E = \mu v^2/2$:

$$\sigma(v) \;\propto\; e^{-2\pi^2 U_c/E} \;\propto\; \exp\left\{-\left(\frac{E_c}{E}\right)^{1/2}\right\} \quad , \quad E_c \;=\; \frac{2\pi^2 \mu Z_1^2 Z_2^2 e^4}{\hbar^2} \tag{31}$$

for a Coulomb potential $U_c \sim Z_1 Z_2 e^2/\lambda$ with $\lambda = h/p$, i.e., $U_c \propto \sqrt{E}$.

• It follows that the reaction rate assumes the form

$$\frac{dn}{dt} \;\propto\; \int_0^\infty s(v)\, \exp\left\{-\left(\frac{E_c}{E}\right)^{1/2} - \frac{E}{kT}\right\} dv \tag{32}$$

for thermal nuclei. Here $s(v) = v^3 \sigma(v)\, \exp\{(E_c/E)^{1/2}\}$ is a slowly-varying function of speed v.

Plot: The Gamow Peak

∗ The convolution of the strong rise of the cross section with energy and the exponential decline of the MB distribution yields a strongly-peaked integrand (denoted the **Gamow peak**). This occurs at $E_{\mathrm{G}} \sim E_c^{1/3}\,(kT)^{2/3}$.

• The total rate depends sensitively on temperature. Integration by the method of **steepest descents** would then yield a rate proportional to

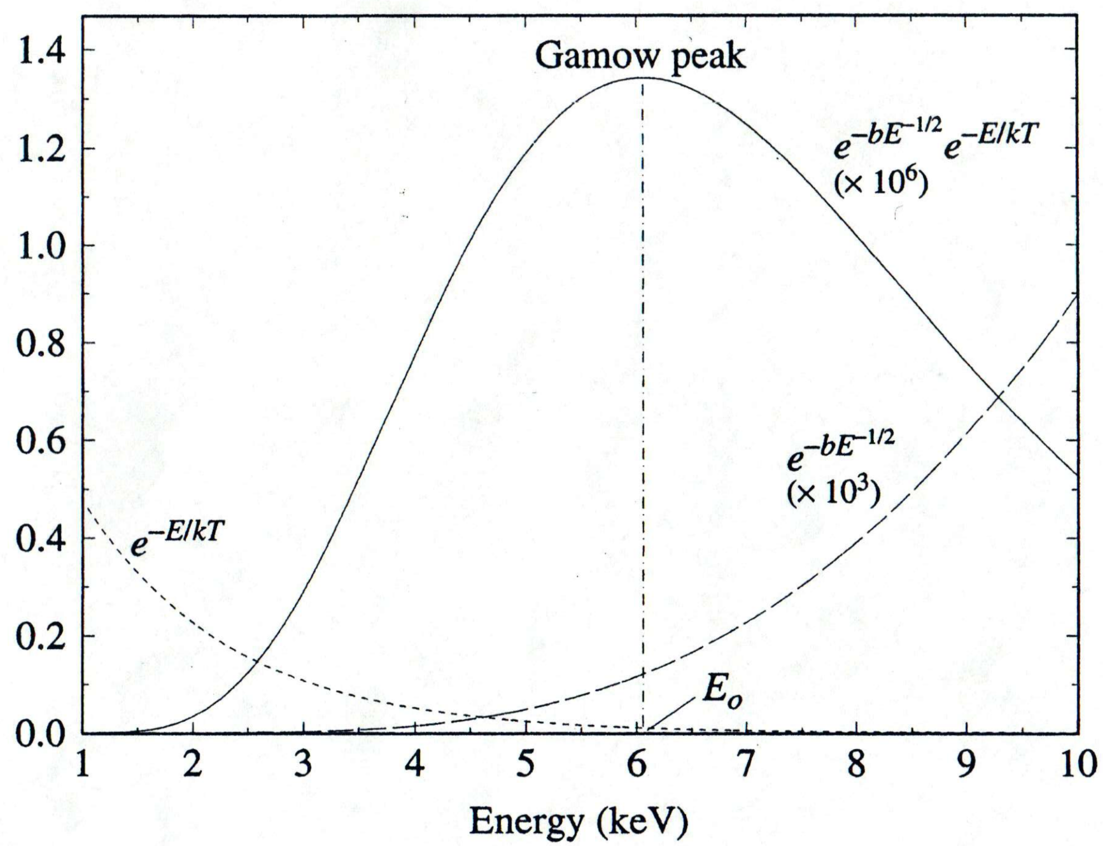
$$\exp\left\{-\left(\frac{E_c}{kT}\right)^{1/3}\right\} \tag{33}$$

Clearly, E_c depends on the composition of the gas. Yet, we can quickly write it down for hydrogen: $E_c = 2\pi^2 \alpha_{\mathrm{f}}^2 m_p c^2 \approx 1\,\mathrm{MeV}$, where $\alpha_{\mathrm{f}} = e^2/\hbar c$.

∗ For the central temperature of the sun, then the Gamow peak is in the 5–10 keV range, and $kT \ll E_c$ underpinning the temperature sensitivity of the rate. This is true also in $M \sim 10 M_\odot$ main sequence stars.

∗ Nuclear interaction rates are complicated by energy resonances, and **electron screening** of nuclear charges in atoms heavier than hydrogen.

Nuclear Interaction Gamow Peak



From: Carroll & Ostlie

- The Gamow peak arises from the competition between the Maxwell-Boltzmann exponential $e^{-E/kT}$ and the Coulomb barrier penetration factor $\exp(-bE^{1/2})$ in thermonuclear interactions.