• The frequency profile for natural broadening can be calculated using a quantum harmonic oscillator model and by taking the Fourier transform of the force equation; the result is a **Lorentz profile**: **Lang, pp. 200–1**

$$F_\nu \;\propto\; \frac{1}{2\pi}\,\frac{\Delta\nu}{(\nu-\nu_0)^2+(\Delta\nu/2)^2} \tag{30}$$

for a line of width $\Delta\nu$ at frequency ν_0. This profile is normalized to unity, and the radiative lifetime $t = 1/\Gamma$ of the states is given by

$$t \;=\; \frac{1}{\Gamma} \;=\; \frac{1}{2\pi\Delta\nu} \quad . \tag{31}$$

Γ is called the *natural line width.*

• **Doppler broadening** is due to the spread in velocities acquired by atoms (or particles) in thermal equilibrium, i.e. defined by the Maxwell-Boltzmann distribution. Here $v_{\rm rms} = \sqrt{3kT/m}$. Some atoms move toward and some move away from an observer. Since $\Delta\lambda/\lambda = \pm v/c$ $(\equiv \Delta\nu/\nu)$ defines the Doppler effect, we anticipate a width due to Doppler broadening of

$$\Delta\lambda_{\rm D} \;\sim\; 2\lambda\frac{v_{\rm rms}}{c} \;=\; \frac{2\lambda}{c}\sqrt{\frac{3kT}{m}} \qquad \text{or} \qquad \Delta\nu_{\rm D} \;\sim\; \frac{2\nu}{c}\sqrt{\frac{3kT}{m}} \quad . \tag{32}$$

For example, since $T = 5770\,$K at the solar surface, the Doppler broadening of the H_α line should be around 0.427 Å, i.e. 3 orders of magnitude higher than the natural broadening.

• Since the Maxwell-Boltzmann distribution is strongly-peaked, Doppler broadening is inherently *Gaussian* around the peak.

∗ Turbulent motions, particularly in tenuous atmospheres of giant stars, can add and sometimes dominate Doppler broadening.

• **Pressure and collisional broadening**. Since orbitals in atoms can be perturbed by collisions, basically due to the polarizing influence of external electric fields, atomic lines can accordingly be broadened. The cumulative statistical effect of large numbers of closely passing ions on an atom's orbitals also leads to line broadening.

These are known as collisional and pressure broadening, respectively, and both depend on the rate of collisions. Taking $\Delta t \approx l/v$ for a mean free path $l = 1/(n\sigma)$, then Heisenberg's uncertainty principle, appropriate to use for pressure broadening, gives

$$\Delta\lambda_{\rm P} \approx \frac{\lambda^2}{\pi c \Delta t} \approx \frac{\lambda^2\, n\sigma}{\pi c}\sqrt{\frac{3kT}{m}} \qquad \text{or} \qquad \Delta\nu_{\rm P} \sim \frac{n\sigma}{\pi}\sqrt{\frac{3kT}{m}} \quad , \tag{33}$$

where $v_{\rm rms} = \sqrt{3kT/m}$ has been used.

The shape of the pressure broadened profile is complicated to derive, but contains a mixture of Doppler-type (Gaussian) and Lorentzian behavior. It is know as a **Voigt** profile: **Lang, pp. 202**

$$F_\nu \propto \frac{1}{\Delta\nu_{\rm D}\sqrt{\pi}}\, H\left(\frac{\Delta\nu_{\rm P}}{4\delta\nu_{\rm D}}, \frac{(\nu-\nu_0)}{\delta\nu_{\rm D}}\right) \quad \text{for} \quad \delta\nu_{\rm D} = \frac{\Delta\nu_{\rm D}}{2\sqrt{\log_e 2}} \quad , \tag{34}$$

where for $\omega = (\nu-\nu_0)/\delta\nu_{\rm D}$,

$$H(r,\omega) = \frac{r}{\pi}\int_{-\infty}^{\infty} \frac{\exp\{-u^2\}}{r^2+(\omega-u)^2}\, du \quad , \tag{35}$$

and $r = \Delta\nu_{\rm P}/4\Delta\nu_{\rm D}$ as a ratio of pressure ($\Delta\nu_{\rm P}$) and Doppler ($\Delta\nu_{\rm D}$) widths.

Plot: Voigt Profile

- The ratio of the Doppler and pressure broadening widths is just formed from the ratio of Eqs. (32) and (33), setting

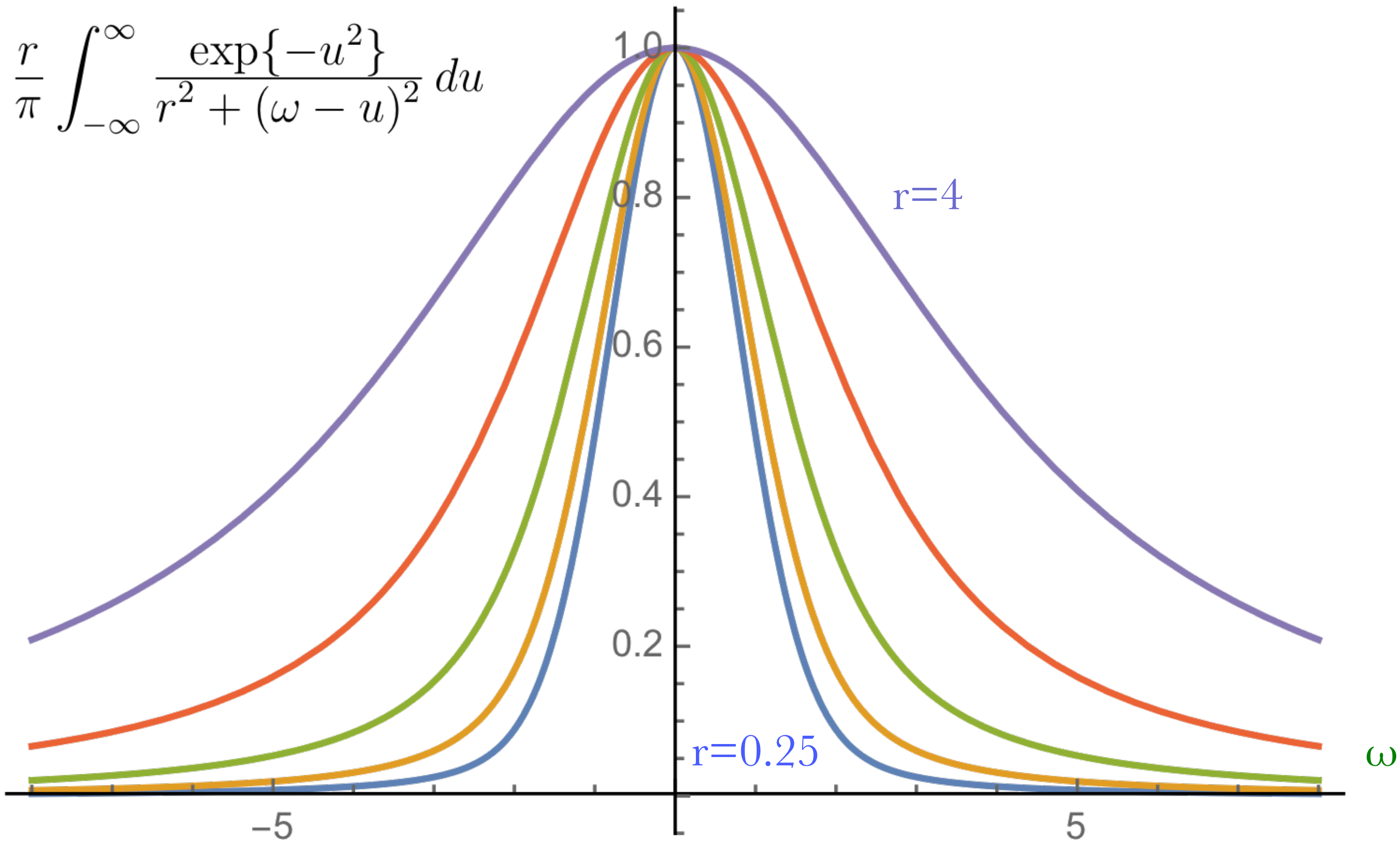
$$\frac{\Delta\lambda_{\rm P}}{\Delta\lambda_{\rm D}} \equiv \frac{\Delta\nu_{\rm P}}{\Delta\nu_{\rm D}} = \frac{\lambda\, n\sigma}{2\pi} \quad . \tag{36}$$

This indicates that *redder lines* in *higher density regions* are subjected to greater pressure broadening.

- The physical basis for the M-K luminosity classification can now be identified: pressure broadening in lines is a key measure of denser versus more tenuous environments. Luminous giant and supergiant stars have lower densities and hence *narrower absorption lines*. Main sequence stars have denser atmospheres and so exhibit *broader lines* due to more frequent collisions.

Voigt Line Profile

$$H(r,\omega) \;=\; \frac{r}{\pi}\int_{-\infty}^{\infty}\frac{\exp\{-u^2\}}{r^2+(\omega-u)^2}\,du$$



- Voigt profiles H(r,ω)/H(r,0) as functions of the frequency ω scaled in terms of the Doppler width. Curves are for different ratios r of the Lorentz width to the Doppler width, r=4,2,1,0.5, and 0.25, from top to bottom.

7. STELLAR STRUCTURE

Matthew Baring – Lecture Notes for ASTR 350, Fall 2021

1 Hydrostatic Equilibrium

While spectral analysis provides powerful clues to the make-up and nature of the sun's outermost layer, we must use more subtle probes of physics to mostly indirectly deduce the nature of its interior. Stars only possess a finite amount of energy, and eventually must die. Yet they survive for a long time, against the relentless pull of gravity.

C & O, Sec. 10.1

• The key property that keeps stars stable against gravitational collapse is pressure, which comes in different varieties, depending on the type of star.

• Consider a cylindrical mass element dm of height dr with its axis oriented radially at some radius r inside the sun.

Plot: Mass Element in a Gravitational Field

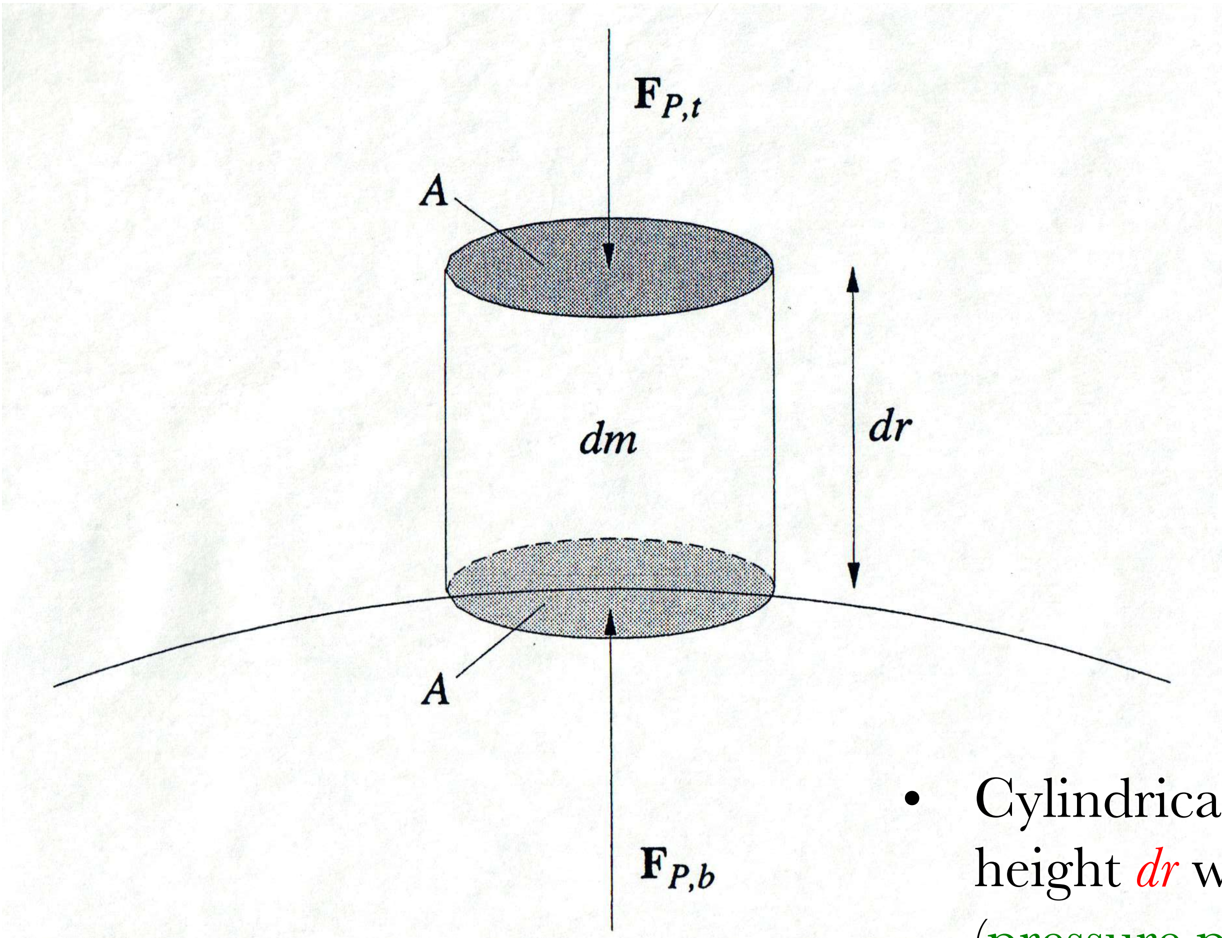
Newton's second law for this element takes the form

$$dm\,\frac{d^2r}{dt^2} \;=\; F_g - \Delta F_P \quad , \tag{1}$$

where F_g is the gravitational force towards the stellar center, and $\Delta F_P = F_{P,b} - F_{P,t} = A\,dP$ is the pressure force differential across the cylinder. If $M(r)$ is the mass interior to the radius, then Newtonian gravity yields $F_g = -GM(r)dm/r^2$. Then the second law becomes:

$$dm\,\frac{d^2r}{dt^2} \;=\; -\frac{GM(r)dm}{r^2} - A\,dP \quad . \tag{2}$$

Mass Element in a Gravitational Field



- Cylindrical mass element *dm* of height *dr* with different forces (pressure plus gravity) exerted on its upper and lower surfaces.

Since $dm = \rho A\, dr$ for no significant density gradient across the element, we can divide through by the volume element $dV = A\, dr$ to derive

$$\rho \frac{d^2 r}{dt^2} = -\frac{GM(r)\rho}{r^2} - \frac{dP}{dr} \quad . \tag{3}$$

Assuming a static star, the acceleration is zero, and we find

$$\boxed{\frac{dP}{dr} = -\frac{GM(r)\rho}{r^2} \equiv -\rho g(r) \quad , \quad g(r) = -\frac{GM(r)}{r^2} \quad .} \tag{4}$$

This is the equation of **hydrostatic equilibrium**.

• This fundamental ingredient of stellar structure implies that a *negative pressure gradient* must exist to counterbalance the attractive force of gravity: i.e. large interior pressures (or temperatures) are requisite.

• e.g. Crudely approximating the pressure gradient by $dP/dr \approx -P_c/R_\odot$, where P_c is the central pressure, since $\rho_\odot = 1.4$ gm/cm^3 on average, we obtain $P_c = GM_\odot \rho_\odot / R_\odot = 3GM_\odot^2/(4\pi R_\odot^4) = 2.7 \times 10^{15}$ dynes/cm^2.

∗ This is two orders of magnitude smaller than the actual value of the solar central pressure, so the functional forms of $M(r)$ and $\rho(r)$ are important ingredients for hydrostatic equilibrium determination.

The *enclosed mass* $M(r)$ can be expressed in terms of the density via a mass conservation relation: $dM(r) = 4\pi \rho r^2\, dr$, i.e.

$$\frac{dM(r)}{dr} = 4\pi r^2 \rho \quad . \tag{5}$$

Therefore, we see that the pressure (and hence temperature) radial structure and the density profile are intimately related.

2 Pressure and Equation of State

It is now timely to make the connection between pressure and temperature. This enters through an **equation of state** (EOS), which essentially couples how a gas can be heated under changes of pressure and/or volume. The simplest EOS is the **ideal gas law** **C & O, Sec. 10.2**

$$\boxed{P_g \; = \; nkT \quad , \quad n \; = \; \frac{N}{V} \quad .} \tag{6}$$

This is the EOS pertinent for the gas component of interiors of main sequence stars; the radiation component contributes pressure too, with a stronger temperature dependence ($\propto T^4$), but only maximizing at about a 5–6% contribution at the solar center.

• For an average mass $\overline{m}$ of a gas particle, we can define $\mu = \overline{m}/m_H$ as the **mean molecular weight**, where $m_H = 1.6735 \times 10^{-24}$ g is the mass of the hydrogen *atom*, so that

$$P_g \; = \; \frac{\rho kT}{\mu m_H} \quad . \tag{7}$$

Given this equation, the equation for hydrostatic equilibrium can be expressed purely in terms of temperature and density.

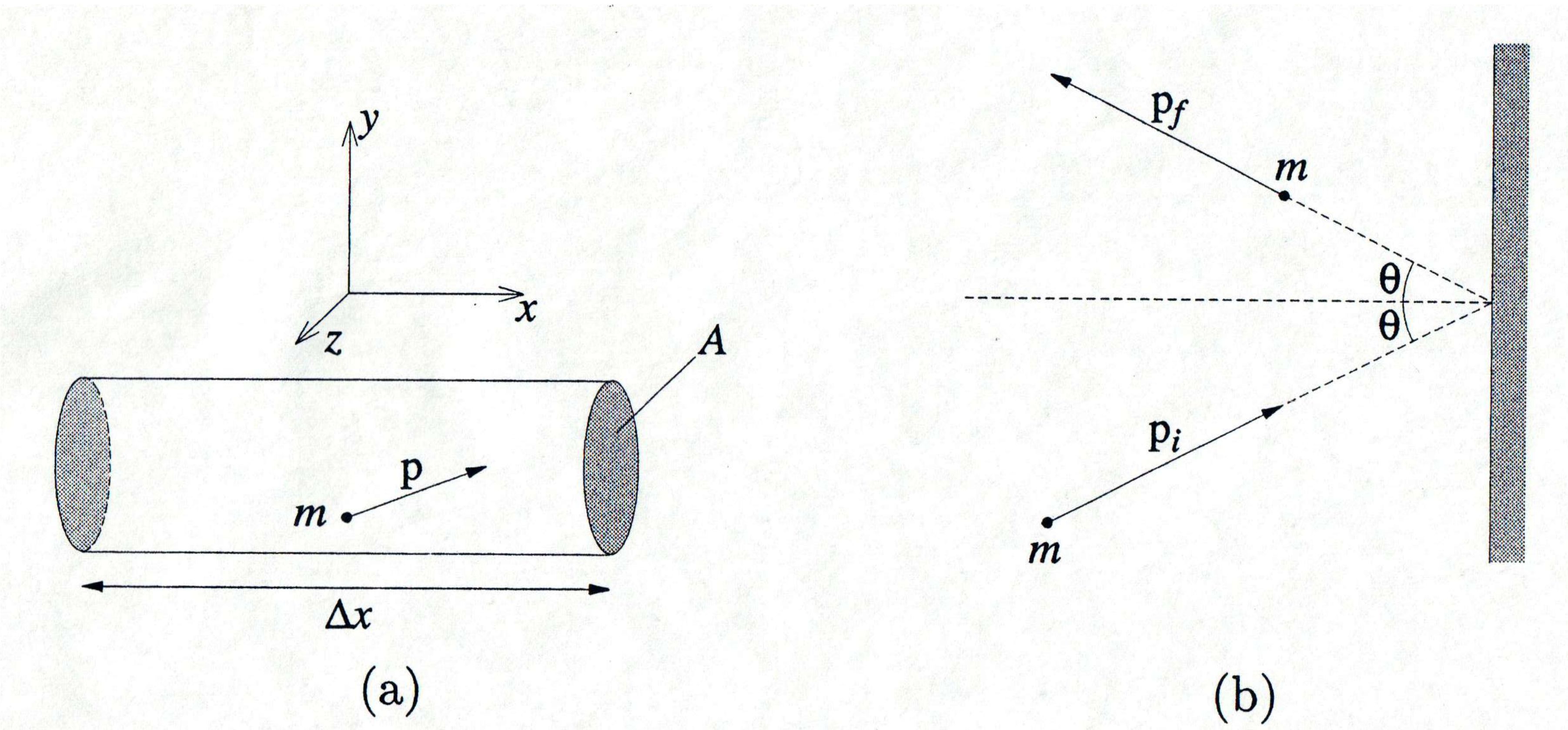
Plot: Ideal Gas Pressure Geometry

It is instructive to derive the ideal gas law. Consider "specular" collisions off a wall, exchanging momentum $-\Delta\mathbf{p} = 2p_x\hat{\mathbf{i}}$ in time Δt: i.e. $\mathbf{f}\Delta t = -\Delta\mathbf{p}$. If Δx is the separation of opposite walls, then the time interval between collisions is $\Delta t = 2\Delta x/v_x$. Hence the force on each wall is of magnitude:

$$f \; = \; \frac{2p_x}{\Delta t} \; = \; \frac{p_x v_x}{\Delta x} \quad . \tag{8}$$

The force is assumed normal to the surface. Since $v^2 = v_x^2 + v_y^2 + v_z^2$, then $p_x v_x = m v_x^2 \to pv/3$ when averaged.

Ideal Gas Pressure Geometry



- Cylinder of gas (a) with atoms/ions of momenta ***p*** that collide and specularly reflect (b) with a wall or surface.

For a number density distribution function n_p such that

$$\frac{N}{\Delta V} = n = \int_0^\infty n_p \, dp \quad , \tag{9}$$

the total force can be written as

$$F = \frac{\Delta V}{3} \int_0^\infty \frac{n_p}{\Delta x} \, pv \, dp \quad , \tag{10}$$

where the volume element is $\Delta V = A \Delta x$. The pressure can then be cast as $P = F/A$, i.e. in the **pressure integral** form

$$P = \frac{1}{3} \int_0^\infty n_p \, pv \, dp \equiv \frac{1}{3} \int_0^\infty m \, n_v \, v^2 \, dv \quad , \tag{11}$$

where $mn_p = n_v$. We can then insert the Maxwell-Boltzmann distribution

$$n_v dv = 4\pi \, n \left(\frac{m}{2\pi kT} \right)^{3/2} \exp\left\{ - \frac{mv^2}{2kT} \right\} v^2 \, dv \quad , \tag{12}$$

and use the integral

$$\int_0^\infty x^4 \exp\left\{ -x^2 \right\} dx = \frac{3\sqrt{\pi}}{8} \tag{13}$$

to yield the *ideal gas law*:

$$\boxed{P_g = nkT} \quad . \tag{14}$$

• e.g. We can use this to estimate the central temperature of the sun. From our previous example, $P_c = 2.7 \times 10^{15}$ dynes/cm^2. This can be set equal to $\rho_\odot kT/\mu m_H$ to yield $T_c = 1.45 \times 10^7\,$K for $\mu = 0.6$ (the value appropriate for complete ionization). This is only 10% lower than the most refined determinations.

• e.g. For these solar values of mean density and molecular weight, radiation pressure exceeds ideal gas pressure when

$$\frac{1}{3} a T^4 > \frac{P_c}{T_c} T \quad \Rightarrow \quad T > 4.2 \times 10^7\text{K} \quad . \tag{15}$$

Hence, more massive main sequence stars require inclusion of radiation pressure in their EOS.