3 The Bohr Atom

Profound developments were made in atomic physics around the end of the 19th century and the beginning of the 20th century. **C & O, Sec. 5.3**

- Thomson discovered the electron in the 1880s;

- Atoms were demonstrated to be neutral, and so contained positive charge;

- Rutherford discovered in 1911, through α-particle scattering, that the positive nucleus occupied only a very small fraction of the atomic volume. For hydrogen, it was determined that $m_p \approx 1836 m_e$.

In 1885, Balmer discovered an empirical relationship for atomic lines in hydrogen:

$$\boxed{\frac{1}{\lambda} = R_{\rm H}\left(\frac{1}{4} - \frac{1}{n^2}\right) \quad ,} \qquad n = 3, 4, 5, \ldots \tag{16}$$

Here $R_{\rm H} = 1.0968 \times 10^5 \ {\rm cm}^{-1}$ is the **Rydberg constant**.

∗ $n = 3 \rightarrow H_\alpha$ Balmer line;

∗ $n = 4 \rightarrow H_\beta$ Balmer line, etc.

- Balmer *predicted* a more general relationship for non-optical lines:

$$\boxed{\frac{1}{\lambda} = R_{\rm H}\left(\frac{1}{m^2} - \frac{1}{n^2}\right) \quad ,} \qquad n = m+1, m+2, m+3, \ldots \tag{17}$$

- Today $m = 1$ denotes Lyman lines, typically in the UV, and $m = 3$ denotes the Paschen series (in IR).

- In classical electromagnetic theory, any orbital model of an $e + p$ atom is basically *radiatively unstable*: the orbiting electron is accelerating, and radiates its energy in 10^{-12} seconds, *generating a continuous spectrum.*

• In 1911, Niels Bohr postulated a semi-classical model of the atom in which *angular momentum was quantized*:

$$\boxed{L \;=\; \frac{nh}{2\pi} \;=\; n\hbar \quad .} \tag{18}$$

The electrostatic force in the atom is given by (in c.g.s.)

$$\vec{F} \;\equiv\; \mu\vec{a} \;=\; \frac{q_1 q_2}{r^2}\,\hat{r} \;=\; -\mu\frac{v^2}{r}\,\hat{r} \tag{19}$$

where the charges can be expressed in units of the electric charge $e = 4.803 \times 10^{-10}\,\text{esu} \equiv 1.602 \times 10^{-19}\,\text{C}$.

Here $\mu = m_e m_p/(m_e + m_p) \approx m_e$ is the reduced mass. Hence, it follows that

$$v \;=\; \frac{e}{\sqrt{\mu r}} \tag{20}$$

is the classical atomic orbital speed of the electron.

• Observe that the kinetic energy in such a model is $K = \mu v^2/2 = e^2/(2r)$, and the potential energy is $U = -e^2/r$, so that the total energy is

$$E_{\text{tot}} \;=\; K + U \;=\; -\frac{e^2}{2r} \;=\; -K \quad , \tag{21}$$

and the bound electrostatic system obeys its own virial theorem!

Now we import Bohr's idea of angular momentum quantization. $L \equiv \mu v r$ is set equal to $n\hbar$ to yield

$$E_{\text{tot}} \;=\; -\frac{e^2}{2r} \;=\; -\frac{1}{2}\,\mu v^2 \;=\; -\frac{1}{2}\,\frac{(n\hbar)^2}{\mu r^2} \quad , \tag{22}$$

implying

$$\boxed{r \;=\; r_n \;\equiv\; \frac{\hbar^2}{\mu e^2}\,n^2 \;=\; a_0\,n^2 \quad ,} \tag{23}$$

where $a_0 = 5.29 \times 10^{-9}\,\text{cm}$ is called the **Bohr radius**; it is the natural atomic scale, and is approximately equal to the reduced electron Compton wavelength divided by $\alpha_{\text{f}} = e^2/(\hbar c)$. *Bohr atomic radii are quantized!*

• The *total energy is then also quantized:*

$$\boxed{E_{\text{tot}} \;=\; -\frac{\mu e^4}{2\hbar^2}\,\frac{1}{n^2} \;\approx\; -\frac{13.6\text{eV}}{n^2} \quad ;} \tag{24}$$

the $n = 1$ case defines the binding energy 13.6eV of hydrogen.

• $\mu e^4/(2\hbar^2)$ is often referred to as the **Rydberg energy**, and is precisely the cutoff energy for the photoelectric effect for hydrogen: i.e., photo-ionization requires UV light.

* n is known as the **principal quantum number** in Bohr's theory.

Transitions between levels $n > m$ lead to photon energies $h\nu = E_n - E_m$, or

$$\boxed{\frac{1}{\lambda} \;=\; R_{\text{H}}\left(\frac{1}{m^2} - \frac{1}{n^2}\right) \quad ,} \tag{25}$$

with

$$R_{\text{H}} \;=\; \frac{\mu e^4}{4\pi\hbar^3 c} \;=\; 1.0968\times 10^5\text{cm}^{-1} \quad . \tag{26}$$

Hence, we have recovered an expression for the Rydberg constant in terms of fundamental constants.

• The excellent agreement between Bohr's model and Balmer's observations was a strong vindication, despite Bohr's ideas not being truly quantum.

• Kirchhoff's laws II and III can now be understood as follows:

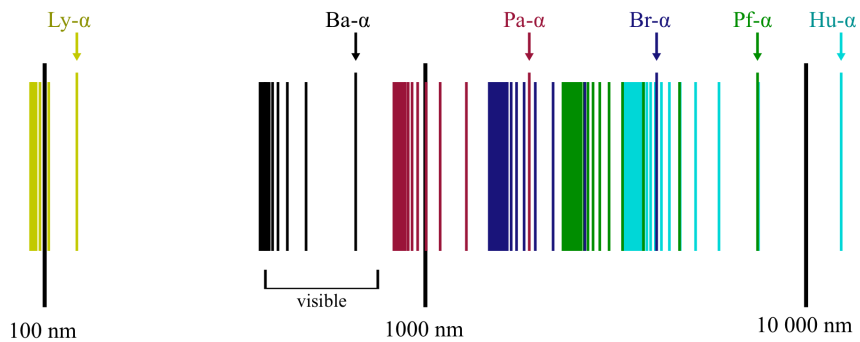
* II: emission lines: $n_{high} \to n_{low}$;

* III: absorption lines: $n_{low} \to n_{high}$.

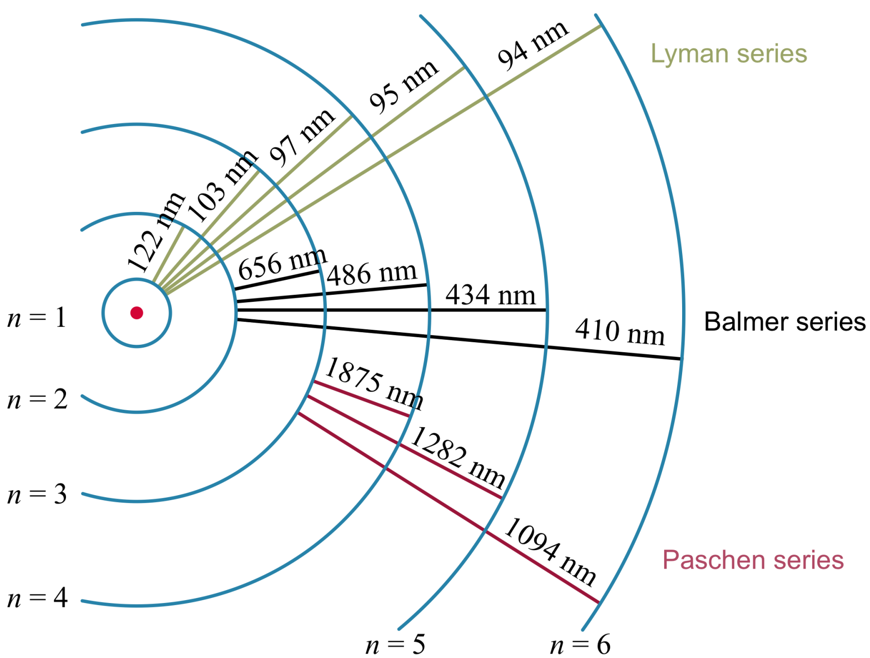
Plot: Bohr atomic transitions

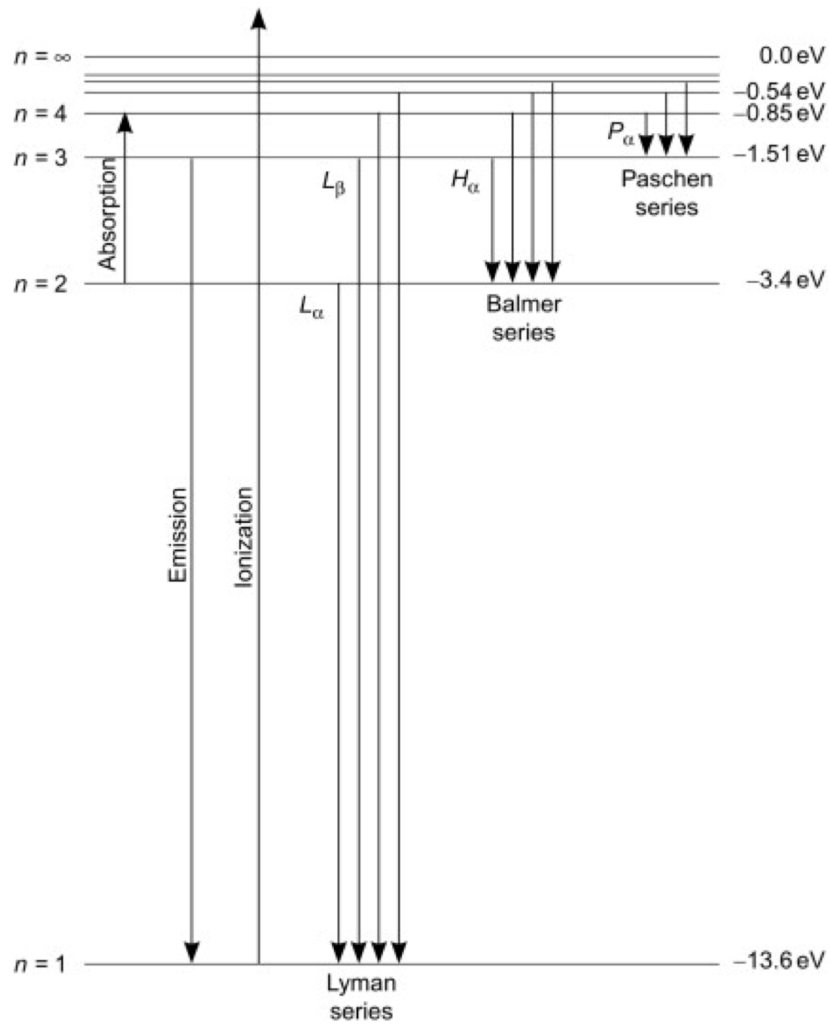
• The impetus for a detailed quantum theory of atoms was mounting at the time of Bohr's enunciation of his model. The culmination was the concept of wavefunctions and the Schrödinger model of hydrogen, a decade later.

Bohr Atom Transitions



Credits: Wikimedia commons + Science Direct





- Energy level and wavelength diagrams for the hydrogen atom.

4 Quantum Concepts

The culmination of Bohr's ideas was the theory of quantum mechanics, which developed in the 1920s and 1930s. The evolution of thought concerning light had moved from particulate nature to wave nature around Newton's time, and then back to particles with the developments by Einstein. This established a **wave-particle duality**. **C & O, Sec. 5.4**

• In his Ph.D. thesis, de Broglie (1927) postulated that matter should also possess such a wave-particle duality, with a particle having frequencies and wavelengths:

$$\nu \; = \; \frac{E}{h} \quad , \quad \lambda \; = \; \frac{h}{p} \quad . \tag{27}$$

The wave nature of matter was eventually verified experimentally by generating an interference pattern using an electron double-slit experiment similar to the Young's set-up for the photon wave property demonstration.

• The **de Broglie wavelength** λ is normally extraordinarily small due to a particle's mass. For elementary particles, λ represents the quantum scale of them, and is only regularly sampled at extremely high densities, around $1/\lambda^3$, which are typically around nuclear densities.

• Neutron stars are sufficiently dense to approach this regime. Taking their scale to be that of a black hole, i.e. a radius around the Schwarzschild radius $R_{\rm S} = 2GM/c^2$, then the ratio of their de Broglie wavelength (for $v \approx c$) to their Schwarzschild radius is

$$\frac{\lambda}{R_{\rm S}} \; = \; \frac{hc}{2GM^2} \tag{28}$$

which is typically much less than unity. Eq. (28) becomes unity when the mass scale and length scale are approximately

$$M_{\rm P} \; = \; \sqrt{\frac{\hbar c}{G}} \; = \; 2.18\times 10^{-5}\,{\rm g} \quad , \quad \lambda_{\rm P} \; = \; \sqrt{\frac{\hbar G}{c^3}} \; = \; 1.62\times 10^{-33}\,{\rm cm} \quad . \tag{29}$$

These are the scales at which gravity and quantum mechanics come together, and define the **Planck mass** and **Planck length**, respectively. **C & O, p. 1234**

The simplest wave description of a particle in constant motion is a plane-wave:

$$\exp\Big\{ i(\vec{k}.\vec{x} - \omega t) \Big\} \;\equiv\; \exp\Big\{ \frac{i}{\hbar}(\vec{p}.\vec{x} - Et) \Big\} \quad . \tag{30}$$

Quantum mechanics builds on this structure for bound systems such as the hydrogen atom, via the Schrödinger dynamical equation.

$*$ Such plane waves are the incoming states for particles in nuclear scattering theory calculations. They are also the initial and final states for free particles and photons in perturbation theory calculations of interactions in **quantum electrodynamics** (QED) and **quantum chromodynamics** (QCD) involving Feynman diagrams.

• The Fourier transform properties of plane waves immediately lead to **Heisenberg's Uncertainty Principle**:

$$\boxed{\Delta x \Delta p \;\gtrsim\; \frac{\hbar}{2} \quad , \quad \Delta t \Delta E \;\gtrsim\; \frac{\hbar}{2} \quad .} \tag{31}$$

between conjugate variables. *It is impossible to localize a particle of finite momentum to a point in space, or vice versa.*

$*$ The quantization of energy levels in an atom ($\Delta E = 0$) leads to total lack of localization in time, i.e. $\Delta t \to \infty$; *precise energy states are stable and of infinite duration.*

$*$ Excited energy states have a natural width or finite lifetime τ so that $\Delta E \sim \hbar/\tau > 0$. These lifetimes are typically very short, $\ll 1\mu\,$sec.

• Matter is described under quantum mechanics by probability distributions, i.e. wavefunctions.

$*$ This permits **quantum mechanical tunneling** through potential barriers, for example repulsive nuclear ones, that are energetically impenetrable in a classical description. This we will explore in our subsequent **thermonuclear reaction** studies of stellar cores.

[*Reading Assignment: C & O pp. 135-6: spin and the Pauli exclusion principle for fermions*]