# 3 Blackbody Radiation

Blackbody radiation is the principal continuum spectrum that is inherently thermal in nature, an idealized form that is never a perfect reality.

• A blackbody absorbs all light incident upon it and re-radiates it with a characteristic spectrum that is determined by quantum mechanics.

## 3.1 Wien and Stefan-Boltzmann Laws

• The blackbody spectrum peaks at a wavelength $\lambda_{\rm max}$ that is given by **Wien's displacement law**: **C & O, Sec. 3.4**

$$\lambda_{\rm max} T \ = \ 0.290 \text{ cm } K \quad . \tag{13}$$

e.g. for $T_\odot = 5.77 \times 10^3$ as the surface solar temperature, Wien's law gives $\lambda_{\rm max} = 0.290/5.77 \times 10^3 \equiv 5030$ Å(Angstroms).

* Hence, hot stars are blue, and cool stars are red.

The total luminosity of a blackbody was shown by Stefan in 1879 to obey

$$\boxed{L \ = \ A\sigma T^4 \quad ,} \tag{14}$$

for $A$ being the surface area of the blackbody. Eq. (14) is known as the **Stefan-Boltzmann Law**, and

$$\sigma \ = \ 5.67 \times 10^{-5} \text{erg sec}^{-1}\text{cm}^{-2} K^{-4} \tag{15}$$

is the Stefan-Boltzmann constant.

• Boltzmann explained Stefan's experimental determination in terms of thermodynamics and electromagnetism.

e.g. For a sphere like the sun, with a radius $R_\odot = 6.96 \times 10^{10}$ cm,

$$L_\odot \ = \ 4\pi R_\odot^2 \sigma T_\odot^4 \ = \ 3.826 \times 10^{33} \text{erg/sec} \ \Rightarrow \ T_\odot \ = \ 5770K \tag{16}$$

is the effective temperature of the solar surface.

## 3.2 Planck Spectrum

The actual shape of the spectrum was explained by Planck, who found an empirical formula for the **intensity**

**C & O, Sec. 3.5**

$$B_\lambda(T) \;=\; \frac{a}{\lambda^5(e^{b/\lambda T}-1)} \quad , \tag{17}$$

where $B_\lambda(T)\,d\lambda\,dA_\perp\,d\Omega$ is the radiation energy per unit area ( $dA_\perp = dA\,\cos\theta$ ), per unit wavelength $d\lambda$ , per unit solid angle $d\Omega$ .

$*$ The units of $B_\lambda(T)$ are erg cm$^{-3}$ sec$^{-1}$ sr$^{-1}$, or often, to avoid unit confusion, erg cm$^{-2}$ Å$^{-1}$ sec$^{-1}$ sr$^{-1}$.

**Plot:** $B_\lambda$ solid angle geometry

• Consider standing waves or *harmonic modes* in a cavity of length $L$ . Planck theorized such a gedanken experiment, where waves that do not interfere with themselves destructively must possess wavelengths

$$\lambda \;=\; \frac{2L}{n} \quad , \quad n \;=\; \text{integer.} \tag{18}$$

Classical thermodynamics could yield an infinite amount of energy for standing waves, due to high-frequency light (the so-called *UV problem*).

• Planck proposed that the waves could only have discrete, integer multiples of a minimum wave energy.

$*$ The quantum of minimum energy was

$$h\nu \;=\; \frac{hc}{\lambda} \quad \Rightarrow \quad \varepsilon_\gamma \;=\; \frac{nhc}{\lambda} \quad . \tag{19}$$

Since the volume scales as $\lambda \propto 1/\nu$ , the phase space density scales as $d^3\nu \propto \nu^2 d\nu$ , so that

$$dN \;=\; \frac{8\pi V}{c^3}\,\nu^2\,d\nu \tag{20}$$

is the number of modes. Here $8\pi \;=\; 2\times 4\pi$ , where 2 is the number of polarizations of the modes.

# The Specific Intensity

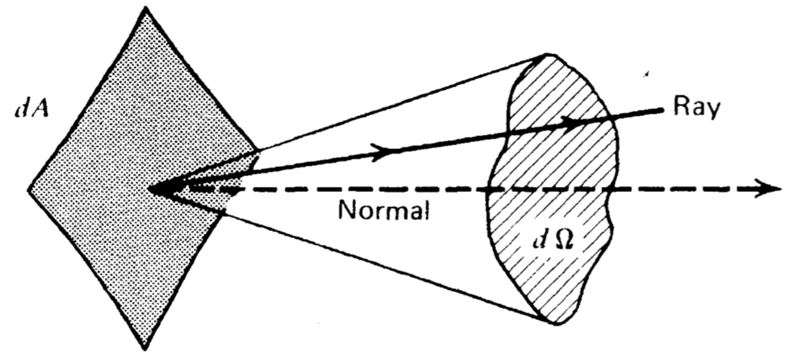



***Figure 1.2*** ***Geometry for normally incident rays.***

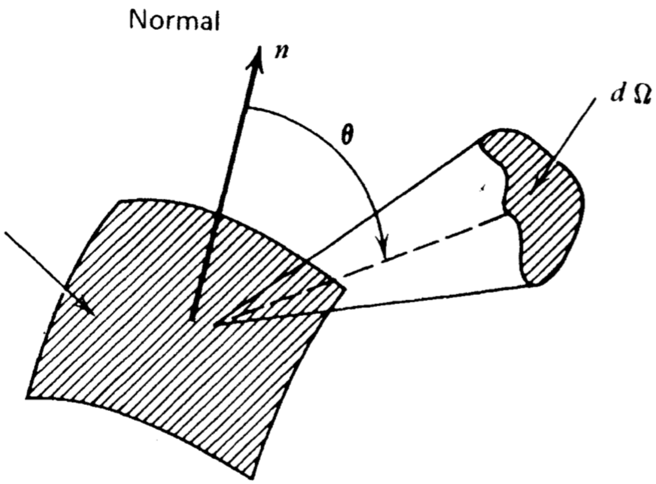



***Figure 1.3*** ***Geometry for obliquely incident rays.***

This must be combined with thermal statistics of a Bose gas: occupation numbers are exponential Boltzmann factors, and are summed over all states:

$$\sum_{n=1}^{\infty} e^{-\varepsilon_\gamma/kT} \;\equiv\; \sum_{n=1}^{\infty} e^{-nhc/\lambda kT} \quad . \tag{22}$$

These ingredients led Planck to the semi-quantum formula:

$$\boxed{B_\lambda(T) \;=\; \frac{2hc^2}{\lambda^5 (e^{hc/\lambda kT} - 1)} \quad .} \tag{23}$$

This is the famous **Planck spectrum** of blackbody radiation. The value of **Planck's constant** $h$ was determined to be

$$h \;=\; 6.626 \times 10^{-27} \text{erg sec} \quad , \tag{24}$$

and the quantum theory of light was established by Einstein's analysis of the photoelectric effect.

In frequency space, the Planck spectrum is

$$\boxed{B_\nu(T) \;=\; \frac{2h\nu^3}{c^2 (e^{h\nu/kT} - 1)} \quad .} \tag{25}$$

$\boxed{\textbf{Plot:}}$ Blackbody spectrum in $\nu$ space

$\ast$ Planck's quantization of light energies fixed the UV problem because it "suppressed" high frequency modes via the Boltzmann factors $e^{-h\nu/kT}$ by ascribing quantum energies $h\nu$ to photons.

- The **Wien limit** is obtained for $h\nu \gg kT$, for which

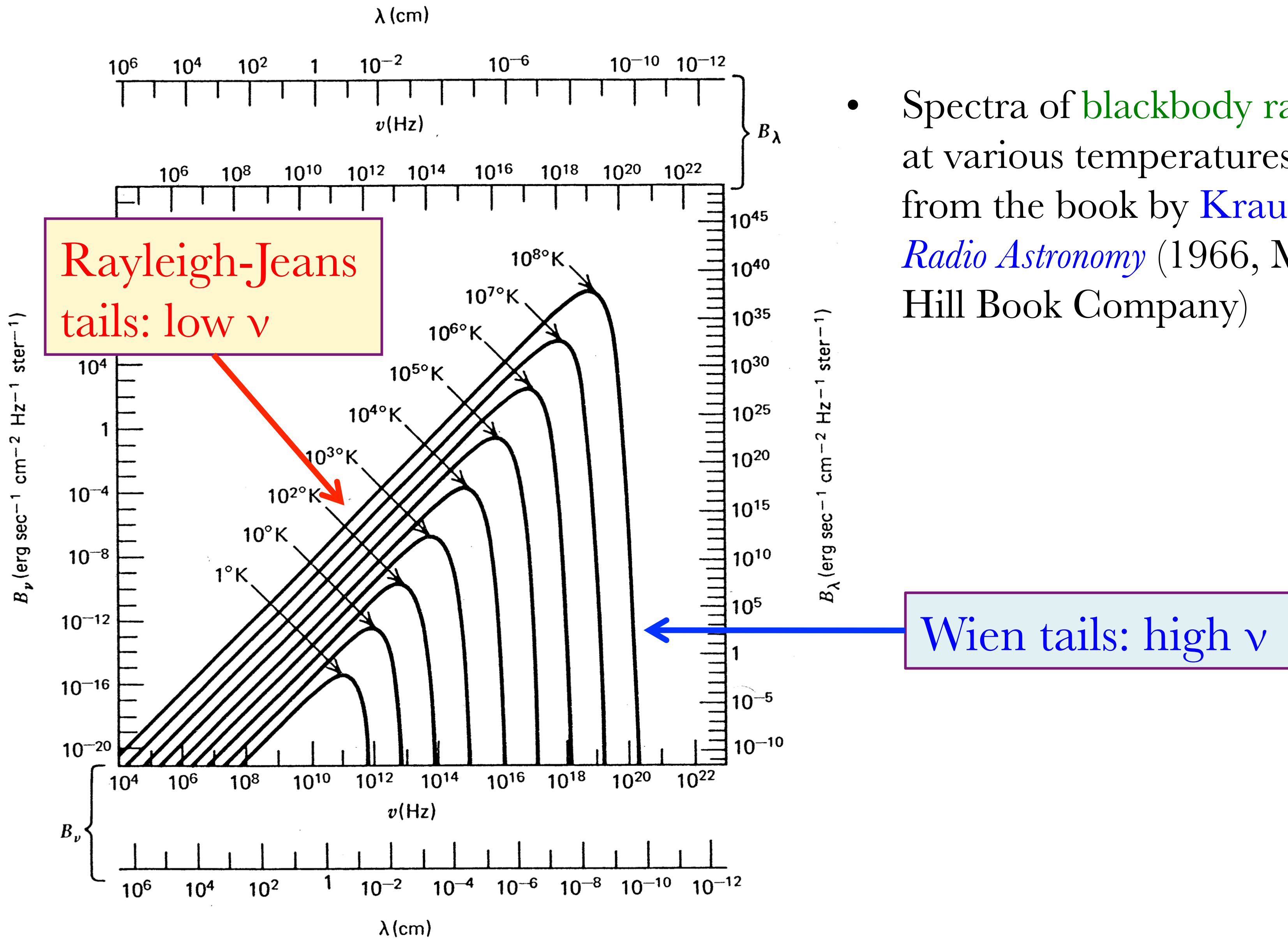
$$B_\nu(T) \;\approx\; \frac{2h\nu^3}{c^2} \exp\left\{ -\frac{h\nu}{kT} \right\} \quad . \tag{26}$$

In the long wavelength **Rayleigh-Jeans** limit, $h\nu \ll kT$, we have

$$B_\nu(T) \;\approx\; \frac{2\nu^2 \, kT}{c^2} \quad . \tag{27}$$

This was first derived by Rayleigh in 1900 using classical physics.

# Planck Spectra: ν Representation




- Spectra of blackbody radiation at various temperatures. Taken from the book by Kraus, J. D. *Radio Astronomy* (1966, McGraw-Hill Book Company)

• What is the peak of $B_\lambda(T)$? Setting $\chi = hc/(\lambda kT)$, we have

$$\frac{d \log B_\lambda(T)}{d\lambda} = 0 \quad \Rightarrow \quad \frac{d}{d\chi}\left\{\frac{\chi^5}{e^\chi - 1}\right\} = 0 \quad \Rightarrow \quad \chi = 4.965 \ . \tag{28}$$

The Wien displacement follows:

$$\lambda_{\max} = \frac{0.2898}{T(^\circ K)} \text{ cm} \quad . \tag{29}$$

• The Planck spectrum integrates to yield the total flux:

$$\mathcal{F} = \pi \int_0^\infty B_\nu(T)\, d\nu = \frac{2\pi h}{c^2} \int_0^\infty \frac{\nu^3\, d\nu}{e^{h\nu/kT} - 1} = \frac{2\pi (kT)^4}{c^2 h^3} \int_0^\infty \frac{\chi^3\, d\chi}{e^\chi - 1} \ . \tag{30}$$

Note that the factor of $\pi$ represents the flux-weighting factor $\cos\theta$ integrated over a hemisphere, giving $2\pi \times 1/2$.

The integral evaluates to $\pi^4/15$, so that the Stefan-Boltzmann law is reproduced, with $\sigma$ expressed in terms of fundamental constants:

$$\mathcal{F} = \sigma T^4 \ , \quad \sigma = \frac{2\pi^5 k^4}{15 c^2 h^3} \equiv \frac{\pi^2 k^4}{60 c^2 \hbar^3} = 5.67 \times 10^{-5} \text{erg sec}^{-1}\text{cm}^{-2} K^{-4} \ . \tag{31}$$

The number density of Planck photons can be obtained by dividing through by $h\nu$ before the integration over frequencies.

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• The determination of stellar spectra leads to an evaluation of the *effective temperature* $T_e$ of a star. Since stellar spectra are not true blackbodies, $T_e$ is that needed to produce $L_*$, and will differ slightly from the true temperatures in stellar photospheres, as will become apparent in due course.

Thus, the Stefan-Boltzmann law $L_* = 4\pi R_*^2 \sigma T_e^4$ can be employed to determine stellar radii, implying that hotter and bigger stars are more luminous.

# 5. RADIATION PROCESSES AND THE QUANTUM ATOM

Matthew Baring – Lecture Notes for ASTR 350, Fall 2021

## 1 Continuum Mechanisms

In astrophysics, there are other sorts of continuum radiation spectra, some of which originate from thermal electrons, yet all of which do not correspond to thermal equilibrium between photons and particles. Examples are: **C & O, pp. 245-7**

* free-bound and bound-free (photoelectric effect) transitions;
* Thomson/Compton scattering of photons by electrons;
* synchrotron radiation in magnetic fields:
* free-free (bremsstrahlung) radiation.

Note that the last three of these arise from accelerating charges; and therefore are predicted by classical electromagnetism. Yet they do have quantum modifications that are mostly needed for compact object studies.

- We now consider each of these in some detail.

## 1.1 Photo-electric Effect

The photo-electric effect (PE effect) is when a photon is absorbed by an atom and expels an electron. It was the first confirmation of quantization of photon energies: explained by Einstein. **C & O, Sec. 5.2**

* $e^-$ assumes a continuous spectrum of energies from zero upwards;

* The maximum kinetic energy $K_{\rm e,max}$ does not depend on the brightness of the light, only on its wavelength;

* each material (metal) has a cutoff frequency $\nu_c$ (or $\lambda_c = c/\nu_c$) below which the PE effect ceases;

• Einstein proposed that light consisted of particles called **photons** with energy quantized according to

$$\boxed{E_\gamma = h\nu = \frac{hc}{\lambda} \quad ,} \tag{1}$$

where $h$ is Planck's fundamental constant.

• Each metal possesses a minimum **binding energy** $\phi$ (also called the **work function**), usually of the order of a few eV, so that

$$\nu_c = \frac{\phi}{h} \quad , \quad \lambda_c = \frac{hc}{\phi} \quad . \tag{2}$$

The maximum kinetic energy of the electron resulting from the PE effect is

$$\boxed{K_{\rm e,max} = E_\gamma - \phi = \frac{hc}{\lambda} - \phi = h\nu - \phi \quad .} \tag{3}$$

Obviously, $K_{\rm e,max} = 0 \Leftrightarrow \nu = \nu_c, \lambda = \lambda_c$ .

• This linear relationship, and the associated cutoff, was the main reason for Einstein's Nobel Prize in 1921.

• Photo-electric absorption is maximized in the ultra-violet and soft X-ray band. It has to be corrected for when performing X-ray spectroscopy in compact objects such as AGNs and neutron stars.

∗ The severity of the source spectral attenuation below around 5 keV is used to measure the integrated **column density** $n_e d$ along the line of sight to the source at distance $d$.

∗ The cross section for the PE effect above the threshold frequency $\nu_c$ scales roughly as $(\nu/\nu_c)^{-7/2}$.

• Since the direction that an ejected electron emerges depends on the photon polarization (i.e. the orientation of its electric field vector), the photo-electric effect can be used as a tool for **X-ray polarimetry**.

**Plot:** Photo-electron Ejection Dependence on Light Polarization

# Polarization in the Photoelectric Effect