5 Binary Orbits

It is appropriate now to define a center of mass (CM) formalism for use in binary gravitational systems. The center of mass position vector is

C & O, pp. 44–47

$$\mathbf{R} \;=\; \frac{\sum_i m_i\,\mathbf{r}_i}{\sum_i m_i} \qquad (21)$$

Let us focus on the two-body case and set the CM to be the origin: $\mathbf{R} = \mathbf{0}$

$$\frac{m_1\mathbf{r}_1 + m_2\mathbf{r}_2}{m_1 + m_2} \;=\; \mathbf{0} \;\; . \qquad (22)$$

Under gravitational interaction, Newton's Third Law indicates this CM position is fixed at all times.

Plot: Binary orbits

If $\mathbf{r}$ is the displacement vector between the two masses, namely $\mathbf{r} = \mathbf{r}_2 - \mathbf{r}_1$,

$$\mathbf{r}_1 \;=\; -\frac{\mu}{m_1}\,\mathbf{r} \quad ; \quad \mathbf{r}_2 \;=\; \frac{\mu}{m_2}\,\mathbf{r} \quad , \qquad (23)$$

for

$$\mu \;=\; \frac{m_1 m_2}{m_1 + m_2} \qquad (24)$$

as the *reduced mass*. Note the harmonic mean identity

$$\frac{1}{\mu} \;=\; \frac{1}{m_1} + \frac{1}{m_2} \quad . \qquad (25)$$

Setting $M = m_1 + m_2$ as the total mass, then two time derivatives of Eq. (22) yields the result (for momentum $\mathbf{P}$)

$$\frac{d\mathbf{P}}{dt} \;=\; M\,\frac{d^2\mathbf{R}}{dt^2} \;=\; \mathbf{0} \;\; , \qquad (26)$$

i.e. the total force on the self-gravitating system is zero, and as a whole it moves with a state of inertia.

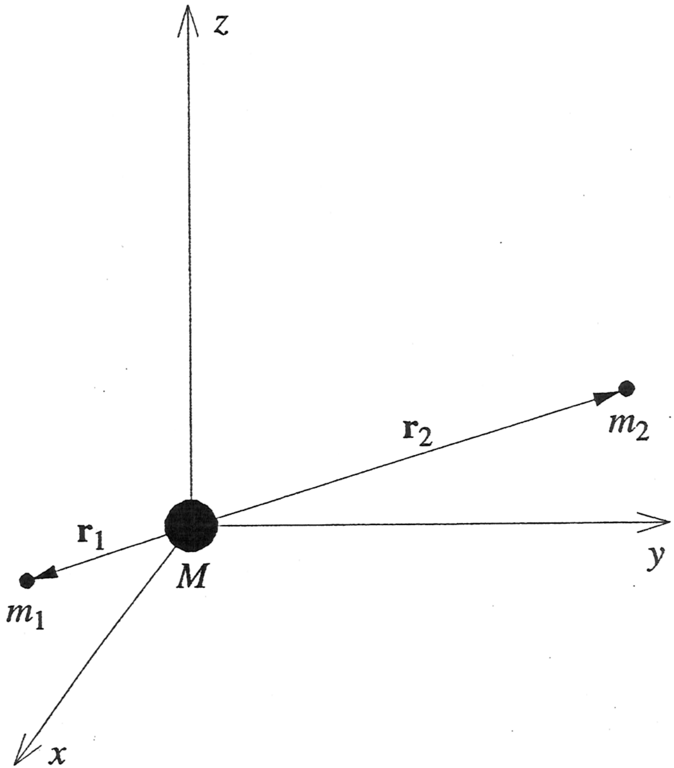

Figure 2.11 The center-of-mass reference frame for a binary orbit, with the center of mass fixed at the origin of the coordinate system.

The total energy distills into a simple form. Setting $\mathbf{v} = d\mathbf{r}/dt$, $v = |\mathbf{v}|$, and $r = |\mathbf{r}_1 - \mathbf{r}_2|$, then using Eq. (23)

$$\frac{m_i v_i^2}{2} = \frac{\mu^2}{2m_i} v^2 \tag{27}$$

for $i = 1, 2$, and

$$\begin{aligned} E &= \frac{1}{2}m_1|\mathbf{v}_1|^2 + \frac{1}{2}m_2|\mathbf{v}_2|^2 - \frac{Gm_1m_2}{|\mathbf{r}_1 - \mathbf{r}_2|} \\ &= \frac{1}{2}\mu v^2 - \frac{GM\mu}{r} \quad . \end{aligned} \tag{28}$$

This is a conveniently elegant form. The force equation is similarly compact for the binary system:

$$\mu \frac{d^2\mathbf{r}}{dt^2} = -\frac{GM\mu}{r^2}\,\hat{\mathbf{r}} \quad , \tag{29}$$

as is the total orbital angular momentum

$$\begin{aligned} \mathbf{L} &= m_1\,\mathbf{r}_1 \times \mathbf{v}_1 + m_2\,\mathbf{r}_2 \times \mathbf{v}_2 \\ &= \mu\,\mathbf{r} \times \mathbf{v} = \mathbf{r} \times \mathbf{p} \quad . \end{aligned} \tag{30}$$

It is left to the student to derive these last two results in entirety.

• Just as before when exploring circular orbits, it is easy to demonstrate that angular momentum is conserved for the two-body motion:

$$\frac{d\mathbf{L}}{dt} = \mu\,\frac{d\mathbf{r}}{dt} \times \mathbf{v} + \mu\,\mathbf{r} \times \frac{d^2\mathbf{r}}{dt^2} = \mathbf{0} \quad . \tag{31}$$

Also, given the constancy of $\mathbf{L}$, the trivial identities $\mathbf{r} \cdot \mathbf{L} = 0 = \mathbf{v} \cdot \mathbf{L}$ establish that the motion is in a plane <u>perpendicular to $\mathbf{L}$</u>.

• These three equations for energy, force and angular momentum clearly indicate that the two-body problem can be reduced to an equivalent one body one, with the total mass M masquerading as the central mass, the separation vector $\mathbf{r}$ acting as space coordinate, and the reduced mass μ taking on the role of the test body mass.

6 Derivation of Kepler's Laws

With all the necessary formalism prepared, we can now proceed to the derivation of Kepler's laws as consequences of Newton's law of gravitation. **C & O, Sec. 2.3**

6.1 Kepler's First Law

This is the most involved of the proofs, and starts by taking the dot product of the force equation (29) with the velocity $\mathbf{v} = d\mathbf{r}/dt$: **Shu, pp. 465**

$$\mathbf{v} \,.\, \frac{d^2\mathbf{r}}{dt^2} \;=\; -\frac{GM}{r^2}\,\mathbf{v}\,.\,\hat{\mathbf{r}} \;=\; -\frac{GM}{r^2}\,\frac{dr}{dt} \quad , \tag{32}$$

since $\mathbf{v} = \hat{\mathbf{r}}\; dr/dt + \mathbf{v}_\theta$, with $\mathbf{v}_\theta . \hat{\mathbf{r}} = 0$. The left side is just half the time derivative of $\mathbf{v}.\mathbf{v} = v^2$, leading to the conservation of energy per unit mass:

$$\boxed{\frac{1}{\mu}\,\frac{dE}{dt} \;\equiv\; \frac{d}{dt}\left\{ \frac{1}{2}\, v^2 - \frac{GM}{r} \right\} \;=\; 0 \quad .} \tag{33}$$

Now we set $v^2 = v_r^2 + v_\theta^2 = (dr/dt)^2 + r^2 (d\theta/dt)^2$ and note that the *specific orbital angular momentum* $\mathcal{J}$ is <u>a constant of the motion</u> (the focus of Kepler's 2nd law below):

$$\mathcal{J} \;\equiv\; \frac{L}{\mu} \;=\; |\mathbf{r}\times\mathbf{v}_\theta| \;=\; r^2\,\frac{d\theta}{dt} \quad , \tag{34}$$

where $\mathbf{L}$ is the orbital angular momentum in Eq. (30). Equation (33) integrates trivially to generate the first order ODE

$$\boxed{\frac{1}{2}\left(\frac{dr}{dt}\right)^2 + \frac{\mathcal{J}^2}{2r^2} - \frac{GM}{r} \;=\; \frac{E}{\mu} \quad ,} \tag{35}$$

where $E < 0$ is the *total energy*, a constant of the motion.

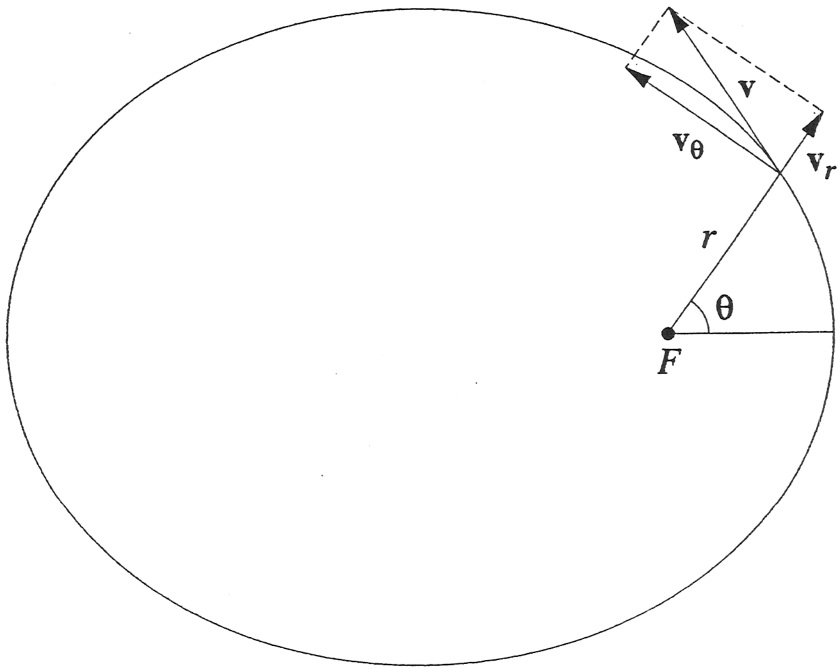

Figure 2.14 The velocity vector for elliptical motion in polar coordinates.

This can be solved for dr/dt and the time eliminated using the angular momentum equation; the result is an equation (with two branches) for the locus of motion:

$$\frac{1}{r^2}\frac{dr}{d\theta} \;=\; \pm\sqrt{\frac{2E}{\mathcal{J}^2\mu}+\frac{2GM}{\mathcal{J}^2 r}-\frac{1}{r^2}} \quad . \tag{36}$$

The substitution $w = 1/r - GM/\mathcal{J}^2$ facilitates solution. One can then write

$$\frac{1}{r^2}\frac{dr}{d\theta} \;\equiv\; -\frac{dw}{d\theta} \;=\; \pm\sqrt{\alpha^2 - w^2} \quad , \quad \alpha^2 \;=\; \frac{2E}{\mathcal{J}^2\mu}+\left(\frac{GM}{\mathcal{J}^2}\right)^2 \quad . \tag{37}$$

Setting $\alpha = GMe/\mathcal{J}^2$ with the **eccentricity** given by Eq. (40) below, the differential equation attains inverse trigonometric form when inverted to find $\theta(w)$. The solution is simply

$$w \;=\; \alpha\cos(\theta-\theta_0) \;=\; \frac{GM}{\mathcal{J}^2}\, e\, \cos(\theta-\theta_0) \;\equiv\; \frac{1}{r}-\frac{GM}{\mathcal{J}^2} \quad , \tag{38}$$

for a constant of integration θ_0, which can be set to zero WLOG. The solution yields the polar equation for an ellipse (for either branch):

$$r \;=\; \frac{a(1-e^2)}{1+e\cos\theta} \quad . \tag{39}$$

The constants of eccentricity and semi-major axis can now be expressed in terms of the conserved quantities of energy and angular momentum:

$$e \;=\; \sqrt{1+\frac{2E\mathcal{J}^2}{G^2M^2\mu}} \quad , \quad a \;=\; -\frac{GM\mu}{2E} \quad . \tag{40}$$

Note that $e < 1$ and $a > 0$ since the total energy is negative.

So we can encapsulate the results as follows: **Kepler's First Law** states that in a binary system, both objects orbit about the center of mass in ellipses, with the center of mass occupying one focus of each ellipse.

$*$ This implies that the sun "wobbles" a little (actually slightly more than its radius), mostly due to the pull of Jupiter.

$*$ The bound orbit solution in Eqs. (39) and (40) can readily be adapted to the case of hyperbolic unbound trajectories such as arise for *interstellar neutrals* incident upon the heliosphere.

6.2 Kepler's Second Law

The total orbital angular momentum in Eq. (30) is conserved in a central force scenario: **C & O, pp. 47 – 49**

$$\frac{d\mathbf{L}}{dt} = \left(\frac{d\mathbf{r}}{dt}\times\mathbf{p}\right)^{0} + \left(\mathbf{r}\times\frac{d\mathbf{p}}{dt}\right)^{0} \equiv \mathbf{0} \quad , \tag{41}$$

since both terms are identically zero. This leads directly to Kepler's second law. The area element inside a curve in 2D polar coordinates can be obtained by approximating a sector with a triangle, and is $dA = (r^2/2)d\theta$. Hence, the rate at which areas are swept out is

$$\frac{dA}{dt} = \frac{1}{2}r^2\frac{d\theta}{dt} = \frac{1}{2}r\,v_\theta \quad . \tag{42}$$

The second equality comes from the polar coordinate form for velocity:

$$\mathbf{v} = \mathbf{v}_r + \mathbf{v}_\theta = \frac{dr}{dt}\hat{\mathbf{r}} + r\frac{d\theta}{dt}\hat{\theta} \quad . \tag{43}$$

The orthogonality of $\mathbf{r}$ and $\mathbf{v}_\theta$ then implies that $rv_\theta = |\mathbf{r}\times\mathbf{v}| = |\mathbf{L}|/\mu$, the angular momentum per unit reduced mass.

Plot: Velocity vector geometry

Hence the conservation of angular momentum immediately leads to **Kepler's Second Law**: the radius vector of an orbiting body under the influence of a central gravitational force sweeps out equal areas in equal times, i.e.,

$$\boxed{\frac{dA}{dt} = \frac{L}{2\mu} = \frac{\mathcal{J}}{2} = \text{const.} \quad , \quad L = |\mathbf{L}| \quad .} \tag{44}$$

• Introducing a third body, even though it might be co-planar, would introduce an extra force vector and thereby break the rotation symmetry of the two-body system, so that angular momentum exchange between it and the binary can arise. This too is the essence of tidal interactions.

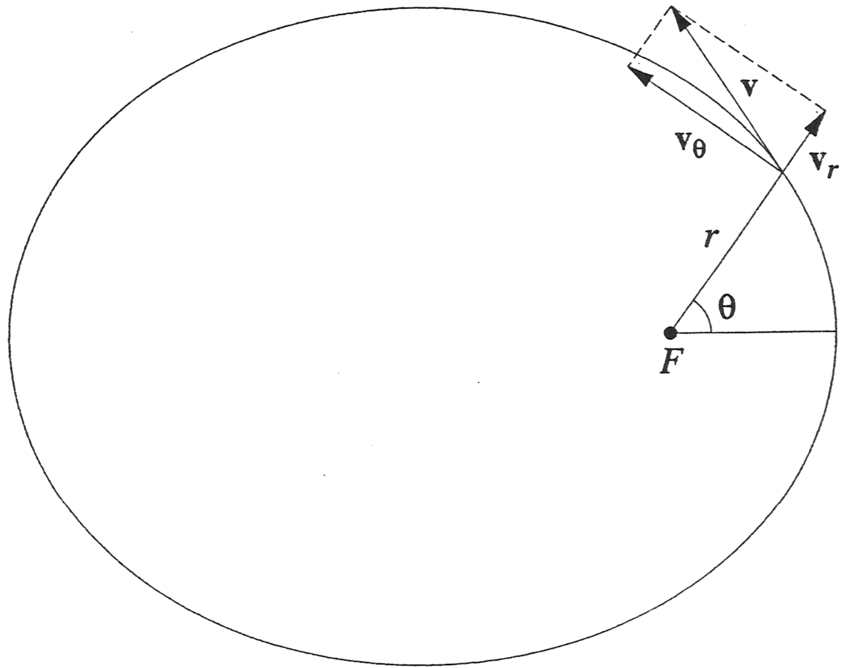

Figure 2.14 The velocity vector for elliptical motion in polar coordinates.

• At perihelion and aphelion, the velocities are purely tangential (circular) so that their values lead to the determination of L in terms of a and e:

$$L \ = \ \mu r_p v_p \ = \ \mu r_a v_a \quad \Rightarrow \quad \frac{v_p}{v_a} \ = \ \frac{r_a}{r_p} \ = \ \frac{1+e}{1-e} \quad . \tag{45}$$

Insertion into the energy equation

$$\frac{1}{2}\mu v_p^2 - G\frac{M\mu}{a(1-e)} \ = \ \frac{1}{2}\mu v_a^2 - G\frac{M\mu}{a(1+e)} \tag{46}$$

which rearranges to

$$\frac{1}{2}\mu v_p^2 \left\{ 1 - \left(\frac{1-e}{1+e} \right)^2 \right\} \ = \ \frac{GM\mu}{a} \left(\frac{1}{1-e} - \frac{1}{1+e} \right) \quad . \tag{47}$$

Multiplying by $\mu r_p^2 = \mu a^2 (1-e)^2$ to generate a result $\propto L^2$ then yields

$$L \ = \ \mu \sqrt{GMa(1-e^2)} \quad , \tag{48}$$

so that for fixed a, the angular momentum is maximized for circular orbits. While this demonstrates the facility of considering orbits at perihelion and aphelion, this result could be simply obtained directly from Eq. (40).