3 Planetary Orbits

There are four key pieces of evidence for the revolution of the Earth about the sun (i.e. proving Copernicus was right), with just retrograde planetary motions dating from before the time of Kepler.

1. Retrograde motions: apparent reversals of direction in the paths of planets.

Plot: inferior and superior planets and retrograde motions

2. Aberration of starlight: (James Bradley, 1729) the direction a telescope points toward a star varies by a small angle $\theta_{\rm aberr}$ on the celestial sphere, mapping out an approximate circle *over the year*. The aberration is due to the finite speed $v_\oplus$ in its revolution (plus the solar peculiar velocity component).

$$\theta_{\rm aberr} \;\approx\; \frac{v_\oplus}{c} \;\sim\; 20.49'' \tag{6}$$

so that $v_\oplus \sim 29.8$ km/sec. This **diurnal variation** (*over the year*) is most distinctive for stars well out of the ecliptic.

$*$ Note that the relevant angular scale is much larger than the sub-arcsecond ones encountered in studying stellar proper motions.

Plot: Aberration of starlight

3. Stellar parallax: an additional *diurnal* variation is that nearby stars move slightly against background distant (and therefore fainter on average) stars.

Plot: Parallactic orbits

Again, this effect is maximized for stars well out of the ecliptic, disappearing for those stars in the ecliptic plane.

$*$ Note that parallaxes are purely geometrical in space, so that the solar peculiar velocity is immaterial to their determination.

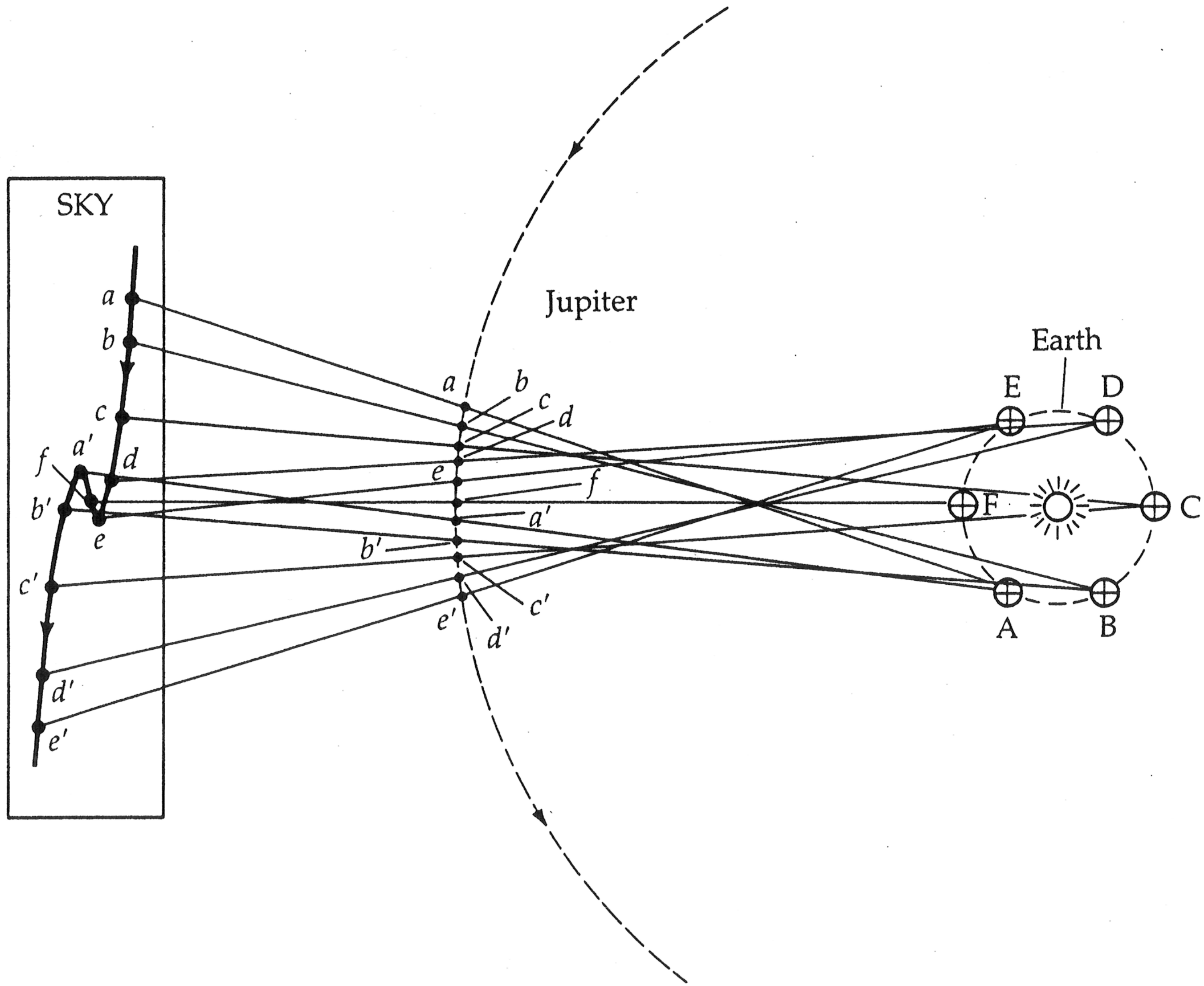


FIGURE 1–3 Retrograde motion in a heliocentric model. As the Earth passes a superior planet, that planet appears to move opposite its normal eastward direction with respect to the stars. Here the Earth passes Jupiter at point *Ff*, which marks the middle of the retrograde motion.

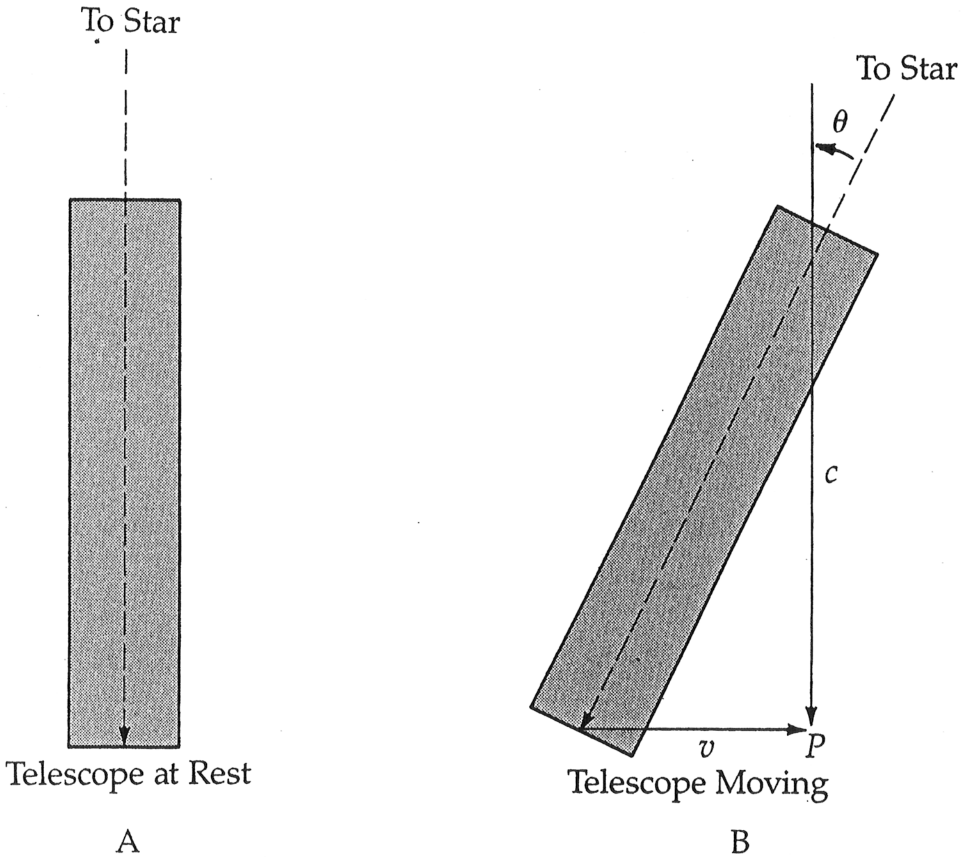


FIGURE 3–11 The aberration of starlight. (A) A telescope is at rest and so pointed upward to observe the light from a star. (B) When a telescope moves at speed v (as a result of the Earth's revolution), it is tilted through an angle θ, so that the starlight reaches P at the same time as the bottom of the telescope does.

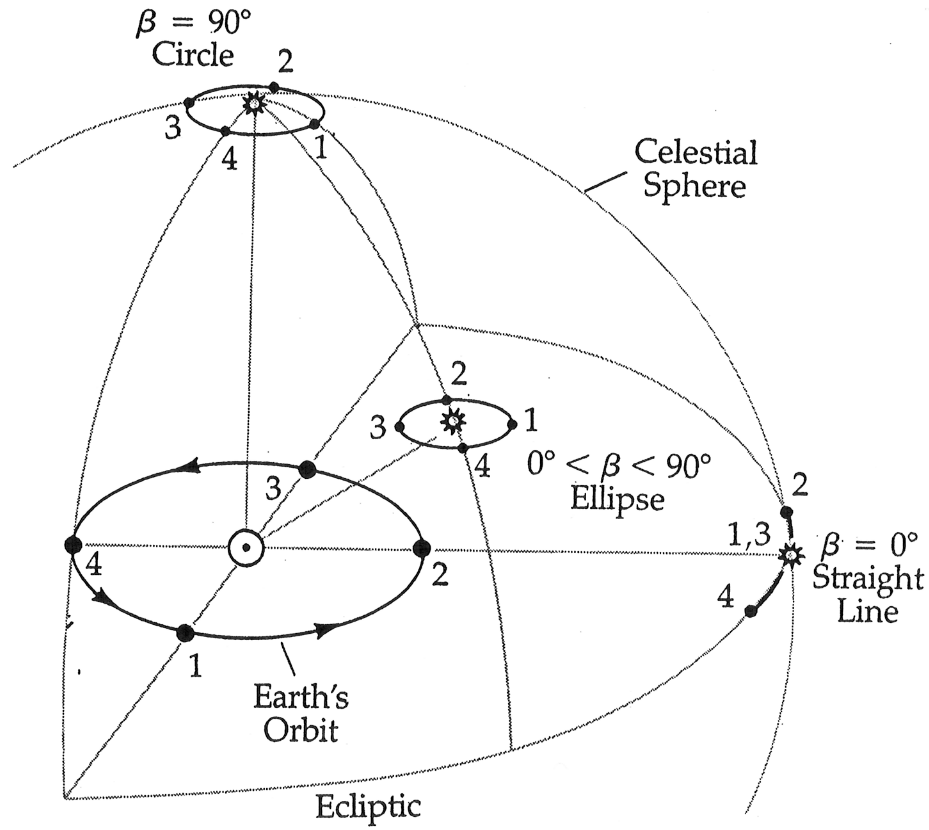


FIGURE 3–12 Apparent stellar orbits resulting from the aberration of starlight. The Earth is shown at four positions in orbit around the Sun (1 to 4). The apparent paths traced out by stars because of the change in the direction of aberration during a year are shown for stars directly above the orbital plane (ecliptic pole), in the plane (the ecliptic), and at an intermediate position on the celestial sphere. The sizes of the apparent orbits are independent of the distances to the stars.

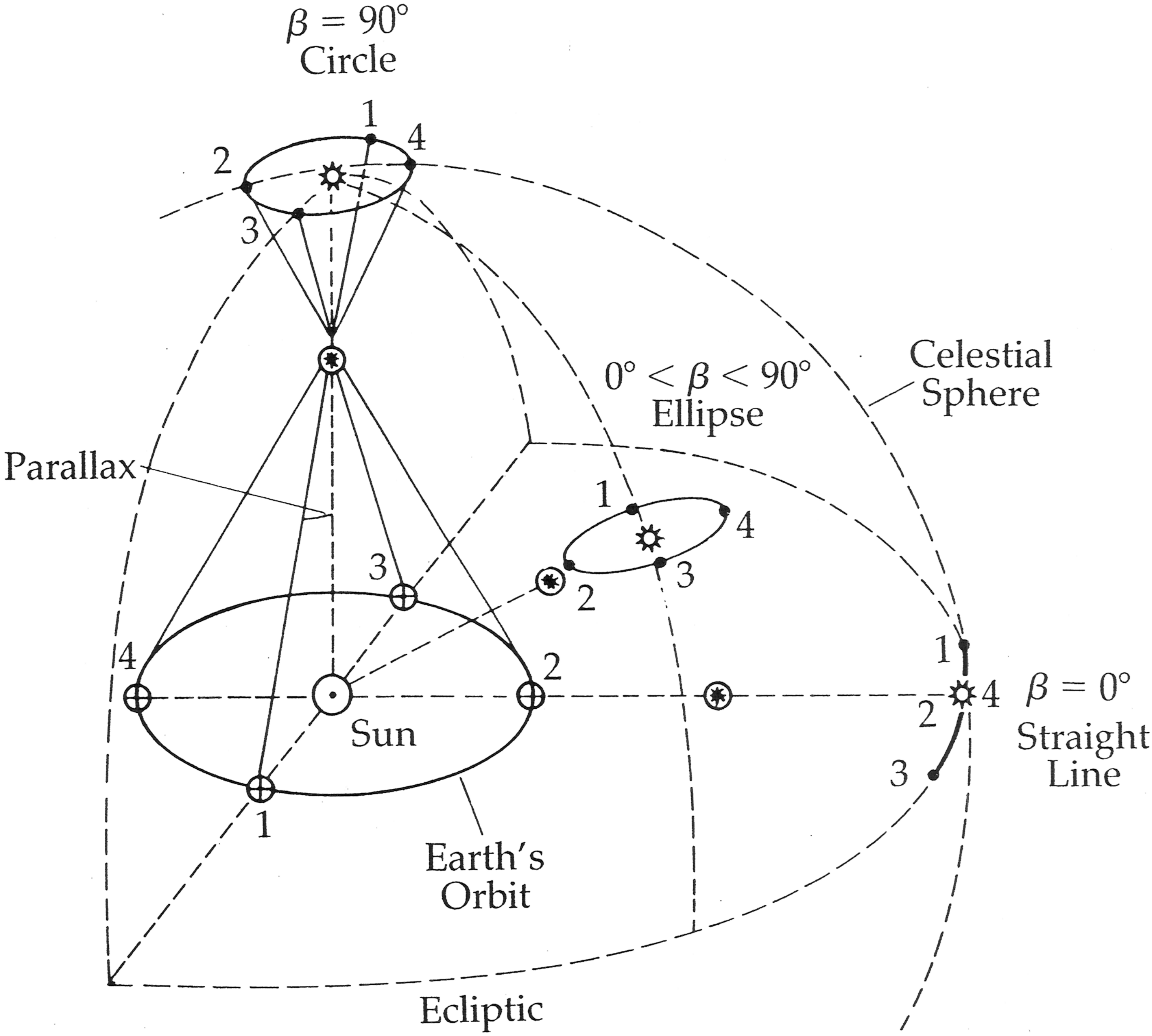


FIGURE 3–14 Parallactic orbits. The Earth's orbital change results in a periodic motion of stellar positions. Note that these are 90° out of phase with respect to the aberration orbits (Figure 3–12) and their angular size depends on the distance to the star. Their shapes depend on their celestial latitude, β.

4. Doppler effect: A truly 20th century concept, this property is evinced in stellar atomic absorption lines:

$$\frac{\Delta\lambda}{\lambda_0} \equiv \frac{\lambda - \lambda_0}{\lambda_0} = \frac{v_\oplus}{c} \sim 10^{-4} \quad , \tag{7}$$

again establishing that $v_\oplus \sim 29.8$ km/sec. Contrary to the previous two indicators, this one is best sampled for stars near the ecliptic plane, where the "radial" Doppler shift has maximum amplitude with a diurnal signal.

4 Newtonian Mechanics

There are four fundamental ingredients in the ensuing discussion of the physics of celestial mechanics due to Newton, namely his three laws of motion and his *Universal Law of Gravitation*. **C & O, Sec. 2.2**

• *Newton's First Law (Inertia):* An object at rest will remain at rest, or an object in motion will remain in motion in a straight line at a constant speed, unless acted upon by an unbalanced force;

• *Newton's Second Law:* the net force acting on an object is proportional to the object's mass m and its resultant acceleration:

$$\mathbf{F} = m\mathbf{a} \equiv m\frac{d\mathbf{v}}{dt} \quad ; \tag{8}$$

• *Newton's Third Law:* for every action (force), there is an equal and opposite reaction.

• To this we add the keystone of gravitational interactions in the cosmos, Newton's *Universal Law of Gravitation*: the gravitational force acting on a mass m at a position vector $\mathbf{r}$, exerted by a mass M at the origin is

$$\mathbf{F}_{grav} = -\frac{GMm}{r^2}\,\hat{\mathbf{r}} \quad , \tag{9}$$

where $G = 6.6726 \times 10^{-8}$ dyne cm^2 g^{-2} is the universal gravitation constant. Here 1 dyne is the cgs unit of force, i.e. that required to accelerate one gram at 1 cm sec^{-2}.

$*$ The inverse square nature of this law is responsible for the elliptic nature of celestial orbits, while the central nature of the vector force guarantees conservation of angular momentum (*invariance under rotations*). These ingredients generate Kepler's laws, as we shall see shortly.

As a step towards this goal, consider the example of a mass $m \ll M$ in circular orbit around mass M, i.e. at any time its velocity vector $\mathbf{v}$ is perpendicular to its position vector r from M. As the orbit sweeps out an angle $d\theta$, so that $\mathbf{v} \to \mathbf{v}' = \mathbf{v} + d\mathbf{v}$, then

$$d\mathbf{v} \;=\; -\hat{\mathbf{r}}\, v\, d\theta \quad , \quad v \;=\; |\mathbf{v}| \quad , \tag{10}$$

since $|d\mathbf{v}| = v\, d\theta$. As $dt = r d\theta / v$, the equation for centripetal acceleration follows:

$$\frac{d\mathbf{v}}{dt} \;=\; -\frac{v^2}{r}\,\hat{\mathbf{r}} \quad . \tag{11}$$

We note that forming a cross product of this equation with $\mathbf{r}$ yields the zero vector, from which we deduce that $\mathbf{r} \times \mathbf{v}$ is a constant vector during orbit:

$$\frac{d}{dt}\left(\mathbf{v} \times \mathbf{r}\right) \;=\; \cancelto{0}{\mathbf{v} \times \mathbf{v}} + \cancelto{0}{\frac{d\mathbf{v}}{dt} \times \mathbf{r}} \;\equiv\; \mathbf{0} \quad , \tag{12}$$

i.e. **angular momentum is conserved.**

$*$ This conservation law is true even of the mass is moving relativistically, as might arise near a black hole, and is a direct consequence of the rotational symmetry of a central force law.

Equating $d\mathbf{v}/dt$ and $\mathbf{F}_{grav}/m$ quickly yields the result $GM/r^2 = v^2/r$. Hence, since the speed is related to the circular orbital period P via $v = 2\pi r/P$, Kepler's third law in the specific case of circular orbits is immediately arrived at:

$$\boxed{\frac{r^3}{P^2} \;=\; \frac{GM}{4\pi^2} \quad .} \tag{13}$$

e.g. As an example, we use this to estimate the mean Earth distance from the sun. Using $G = 6.6726 \times 10^{-8}$ dyne cm^2 g^{-2}, the solar mass $M_\odot = 1.99 \times 10^{33}$ g, and the period $P = 1\,\text{yr} = 3.156 \times 10^7\,\text{sec}$, we have

$$r_\oplus \;=\; \left(\frac{GM}{4\pi^2}\, P^2 \right)^{1/3} \;=\; 1.496 \times 10^{13}\ \text{cm} \tag{14}$$

as the *astronomical unit* (AU). Hence, the light travel time to the Earth from the sun is around 8 minutes.

∗ Note that astronomers do not use $M_\odot$ to determine the AU — instead, they used time delays around the surface of the Earth for the Venus transit of the sun to determine $r_\oplus$ and then infer the solar mass.

Kepler's Third Law provides a powerful means of mass determination by monitoring orbital dynamics in self-gravitating systems.

Plot: Mass determinations at the Galactic Centre: Nobel Prize 2020

e.g. Determination of typical satellite orbital periods around Earth:

$$\begin{aligned} P_{\text{orbit}} \;&\sim\; \left(\frac{GM_\oplus}{4\pi^2\, R_\oplus^3} \right)^{-1/2} \\ &=\; \left(\frac{6.67 \times 10^{-8}\,\text{cm}^3\,\text{g}^{-1}\,\text{sec}^{-2}\ (5.97 \times 10^{27}\text{g})}{4\pi^2\,(6.38 \times 10^{8}\text{cm})^3} \right)^{-1/2} \\ &=\; 5074\ \text{seconds} \;\approx\; 85\ \text{minutes} \quad . \end{aligned} \tag{15}$$

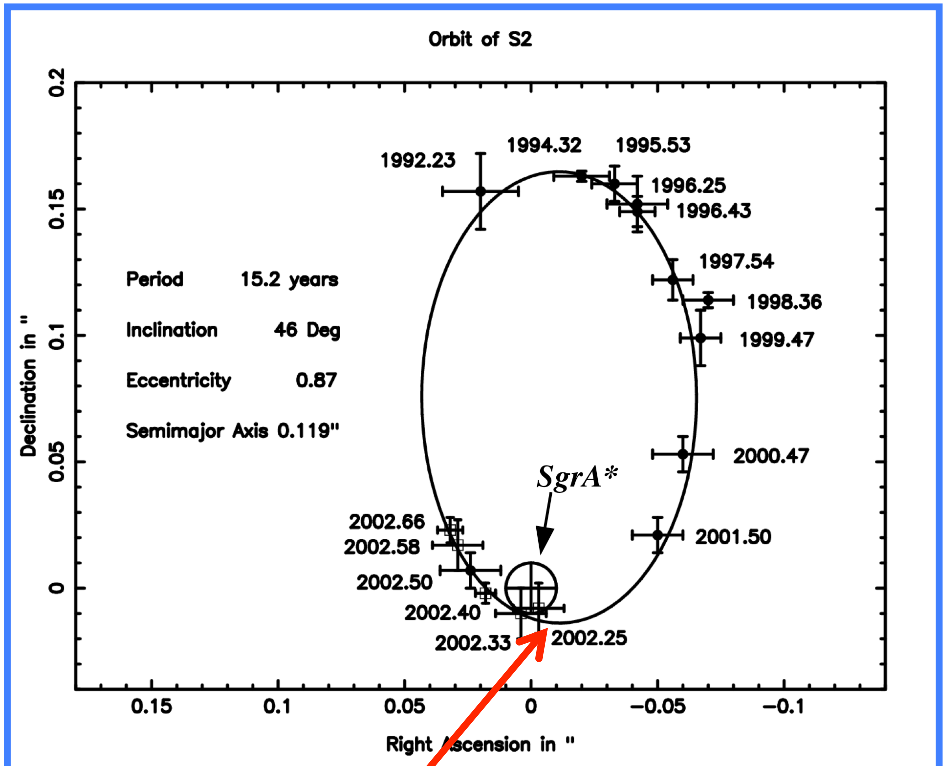
e.g. Orbital velocity of satellites around Earth:

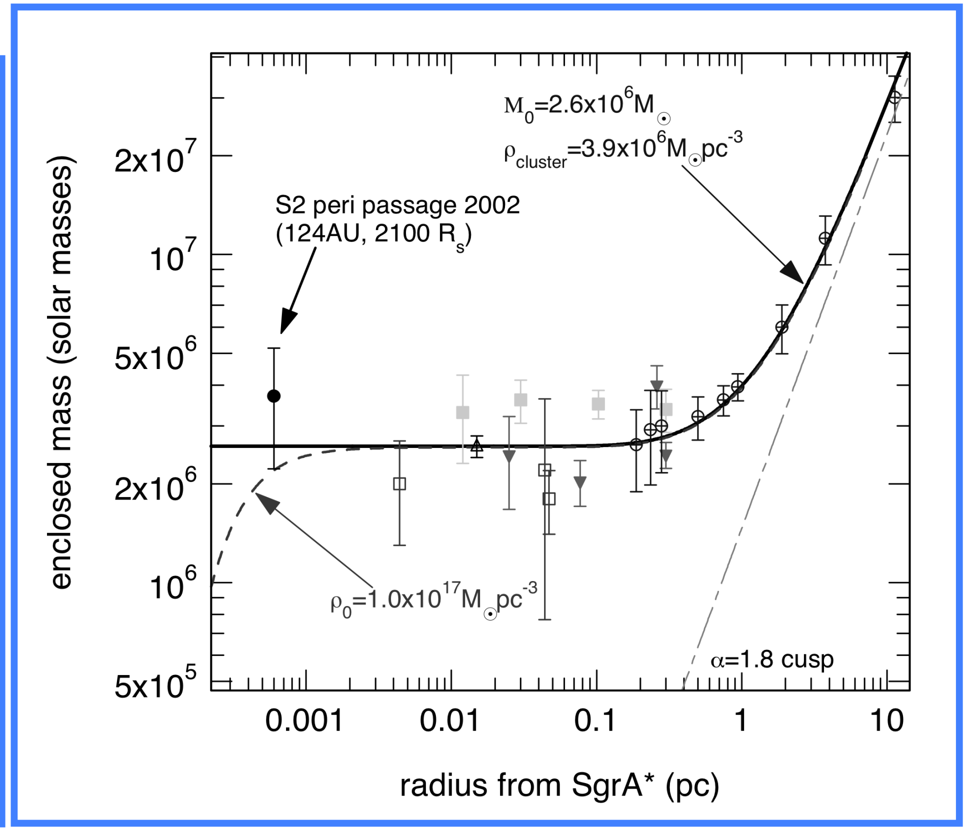
$$\begin{aligned} v_{\text{orbit}} \;&=\; \frac{2\pi R_\oplus}{P_{\text{orbit}}} \;=\; \frac{2\pi\ (6.38 \times 10^{8}\text{cm})}{5074\ \text{sec}} \\ &=\; 7.9\ \text{km/sec} \;\equiv\; 17,760\ \text{miles/hour} \quad . \end{aligned} \tag{16}$$

Reading assignment: Example 2.2 from C & O. The gravity exerted by an extended object is equivalent to that if all the mass is concentrated at its center of mass. N. B. This couples to **Green's functions** in the theory of partial differential equations, solving Poisson's equation for gravity.

Stellar Passages Near the Galactic Centre

Schoedel et al. (2002, Nature 419, 694)





- *Left Panel*: passage of the star S2 in the epoch 1992-2002, which has a Keplerian period of 15 years, imaged in IR using ESO's VLT in Chile, capturing **peribothron** (pericentre) portion cleanly.
- Pericentre offset of 17.2 light hours (124 AU) rules out the presence of *fermion balls* and promotes the existence of a central supermassive black hole (SMBH) of mass 3.7 x $10^6 M_\odot$.
- *Right Panel*: enclosed mass distribution and the S2 data point favoring a SMBH for Sgr A*.

4.1 Potential and Kinetic Energies

Suppose we move a test mass (i.e. planet) from a radius r_i to a radius $r_f > r_i$ in a central gravitational potential. Then the work done is **C & O, pp. 40–42**

$$\begin{aligned} \Delta U \; = \; U_f - U_i \;\; &= \;\; -\int_{r_i}^{r_f} \mathbf{F}_{grav} . d\mathbf{r} \\ &= \;\; \int_{r_i}^{r_f} \frac{GMm}{r^2}\, dr \\ &= \;\; -GMm \left(\frac{1}{r_f} - \frac{1}{r_i} \right) \;\; . \end{aligned} \tag{17}$$

Hence we identify

$$\boxed{U \; = \; -\frac{GMm}{r}} \tag{18}$$

as the gravitational *potential energy*. Since $K = mv^2/2$ is the body's kinetic energy, the total energy is

$$E \; = \; U + K \; = \; \frac{1}{2}mv^2 - \frac{GMm}{r} \quad . \tag{19}$$

This quantity is conserved in orbital dynamics.

N. B. To escape a gravitational pull, we need $E > 0$, which implies $v > v_{esc} = \sqrt{2GM/r}$. Hence, the escape velocity for rockets from Earth is

$$v_{esc} \; = \; \sqrt{\frac{2GM_\oplus}{R_\oplus}} \; = \; 11.2 \text{ km/sec} \;\; . \tag{20}$$

This is just a factor $\sqrt{2}$ larger than the orbital satellite speed.