1. CELESTIAL MECHANICS

Matthew Baring – Lecture Notes for ASTR 350, Fall 2021

1 Kepler's Laws (1609)

Celestial mechanics within and outside the solar system are governed by three key **Kepler's Laws** of orbital motion: **C & O, Sec. 2.1**

- I. Planets move about the sun in elliptical orbits with the sun at one focus.
- II. The radius vector of a planet sweeps out equal areas in equal times.
- III. If a is the semi-major axis, and P is the orbital period, then

$$\frac{a^3}{P^2} \; = \; \text{constant} \tag{1}$$

$\boxed{\textbf{Plot:}}$ Equal area elliptical diagram

$\boxed{\textbf{Plot:}}$ Kepler III: observational evidence

Q. Why are these laws valid?

A. Conservation of energy and angular momentum in an inverse square (gravitational) 2 body force law.

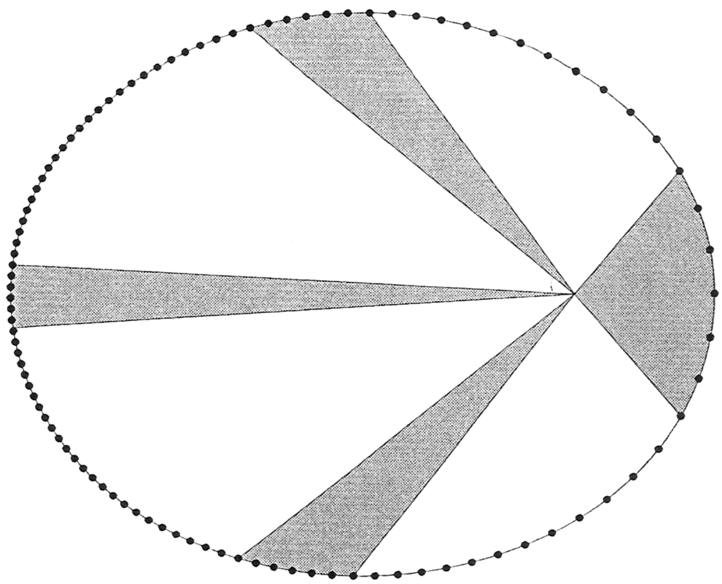

Figure 2.2 Kepler's second law states that the area swept out by a line between a planet and the focus of an ellipse is always the same for a given time interval, regardless of the planet's position in its orbit. The dots are evenly spaced in time.

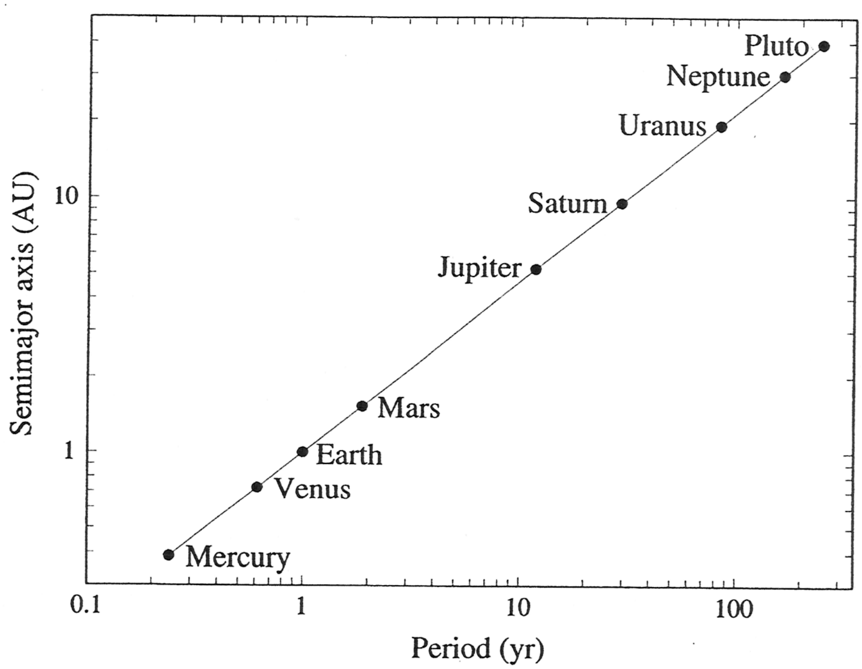

Figure 2.3 Kepler's third law for planets orbiting the Sun.

N.B. Three body Newtonian perturbations (e.g. due to remote planets or planetary oblateness) and general relativistic (GR) perturbations destroy the $1/r^2$ force law symmetry and introduce $1/r^3$ corrections.

$\ast$ e.g. 1: solar oblateness: effect on Mercury's orbit

$\ast$ e.g. 2: presence of Jupiter perturbs Mercury's orbit

$\ast$ e.g. 3: GR corrections due to solar mass precesses end point of semi-major axis (perihelion); a classic test of Einstein's theory circa 1920.

2 Conic Sections

An exploration of the properties of the ellipse, a conic section, provides background to observational measurements of orbital motions.

An ellipse, defined by $|r + r'| =$ constant, possesses a semi-major axis of length a, a semi-minor axis of length b, a principal focus at F (e.g., where the sun is located), and alternate focus F', and has an eccentricity e. **C & O, pp. 28–31**

Plot: Ellipse: shape and definitions

In general, $0 < e < 1$ for an ellipse, with $e = 0$ for a circle, and $e \lesssim 1$ for a cigar-shaped form.

[*Handout: Derivation of Cartesian Form for Ellipse*]

• Converting to polar coordinates, $y = r\sin\theta$ and $x = ae + r\cos\theta$ about the principal focus F. Inserting these into the Cartesian form for the equation of an ellipse yields

$$\begin{aligned} \frac{x^2}{a^2} + \frac{y^2}{a^2(1-e^2)} &= \frac{(ae + r\cos\theta)^2}{a^2} + \frac{r^2\sin^2\theta}{a^2(1-e^2)} = 1 \\ e^2 + \frac{2er}{a}\cos\theta + \left(\frac{r}{a}\right)^2\cos^2\theta + \frac{r^2(1-\cos^2\theta)}{a^2(1-e^2)} &= 1 \quad . \end{aligned} \tag{2}$$

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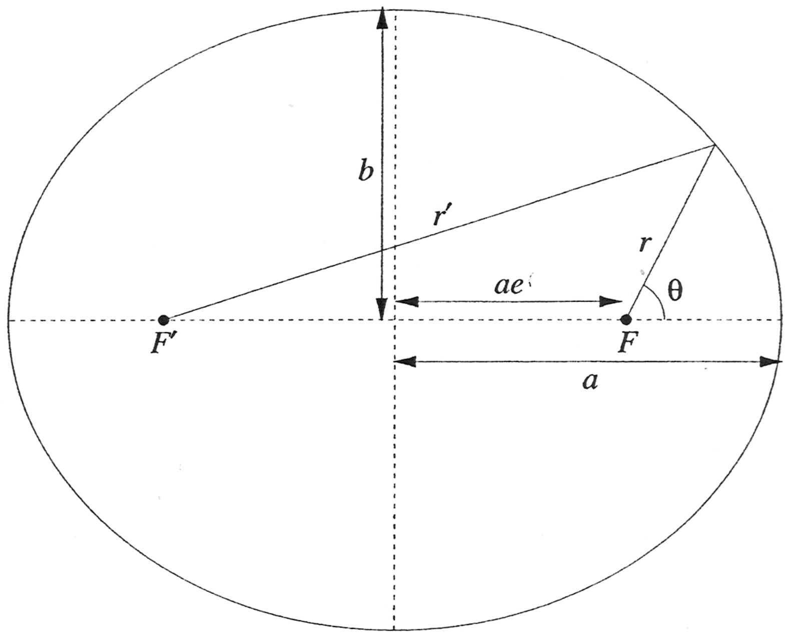

Figure 2.4 The geometry of an elliptical orbit.

Setting $\chi = r/[a(1-e^2)]$, this develops into the quadratic (do this)

$$\chi^2(1 - e^2\cos^2\theta) + 2e\chi\cos\theta - 1 = 0 \tag{3}$$

after some cancellation, whose positive solution yields

$$\boxed{r \;=\; \frac{a(1-e^2)}{1+e\cos\theta} \quad .} \tag{4}$$

Note that $e \to 1$ yields a parabola with $r \propto 1/(1+\cos\theta)$.

$*$ Hyperbolae are defined by $|r - r'| =$ constant. The $e > 1$ branch of solutions corresponds to the hyperbola: $r = a(e^2-1)/(1+e\cos\theta)$.

Plot: Conic sections

Physical interpretation of these conic sections in the context of orbital motions are as follows:

$*$ *elliptical* orbits correspond to bound orbits, such as for satellites, comets, binary stars, and stars in proximity to the Galactic Center.

$*$ *parabolic* orbits are marginally bound/unbound,

$*$ *hyperbolic* trajectories correspond to unbound orbits, applicable to rockets that escape the Earth's gravity, and non-capturing stellar collisions.

Plot: Planetary orbits

• e.g. Mars on 8/27/2003, achieved its nearest approach to Earth in around 60,000 years. For Mars, $a = 1.5237\,\text{AU}$ and $e = 0.0934$, establishing radii to perihelion and aphelion (from the sun) of

$$\begin{aligned} r_p &= a(1-e) = 1.3814 \text{ AU} \\ r_a &= a(1+e) = 1.6660 \text{ AU} \end{aligned} \tag{5}$$

This yields around a 20% change in radius from the sun during its orbit. However, since the Earth is at 1 AU, the mean distance between Mars and Earth is typically around $2.5-3$ AU, but the closest approach corresponds to 0.38 AU. *i.e. Mars was very bright on 8/27/2003* !

$\boxed{\textbf{Plot:}}$ Conic sections

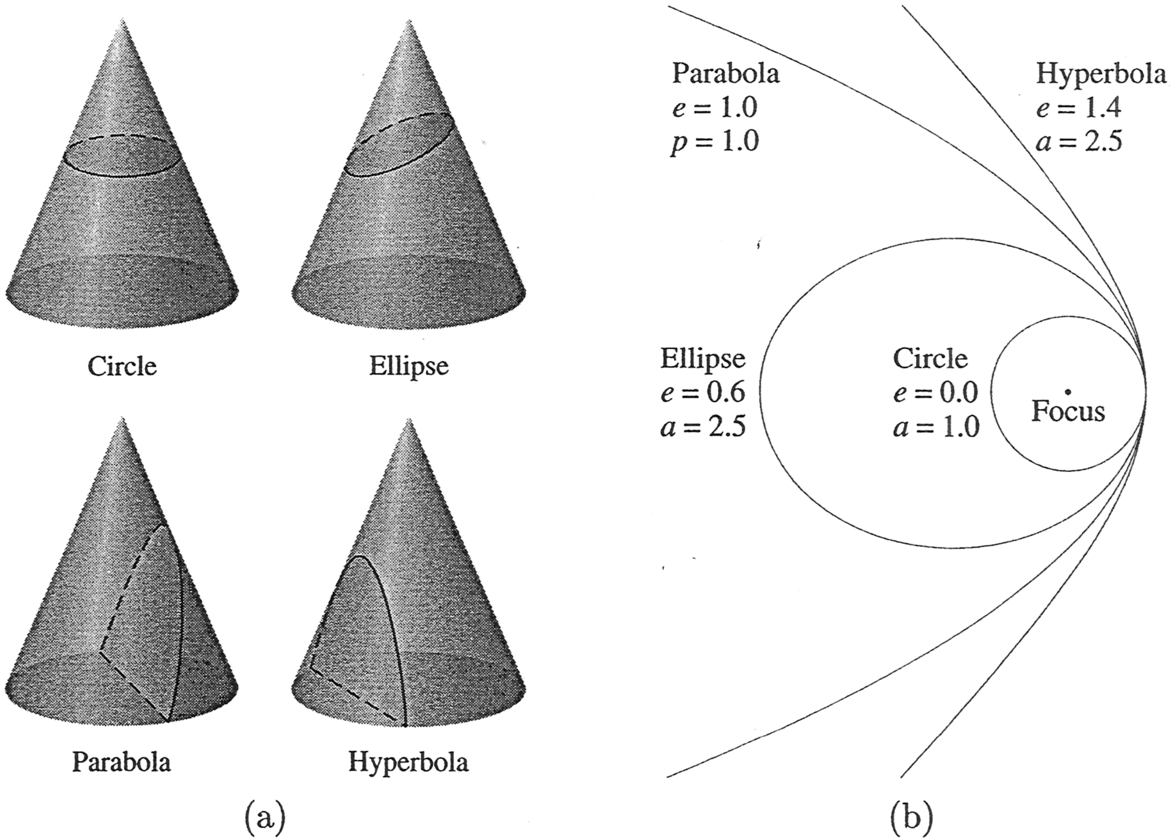

Figure 2.5 (a) Conic sections. (b) Related orbital motions.

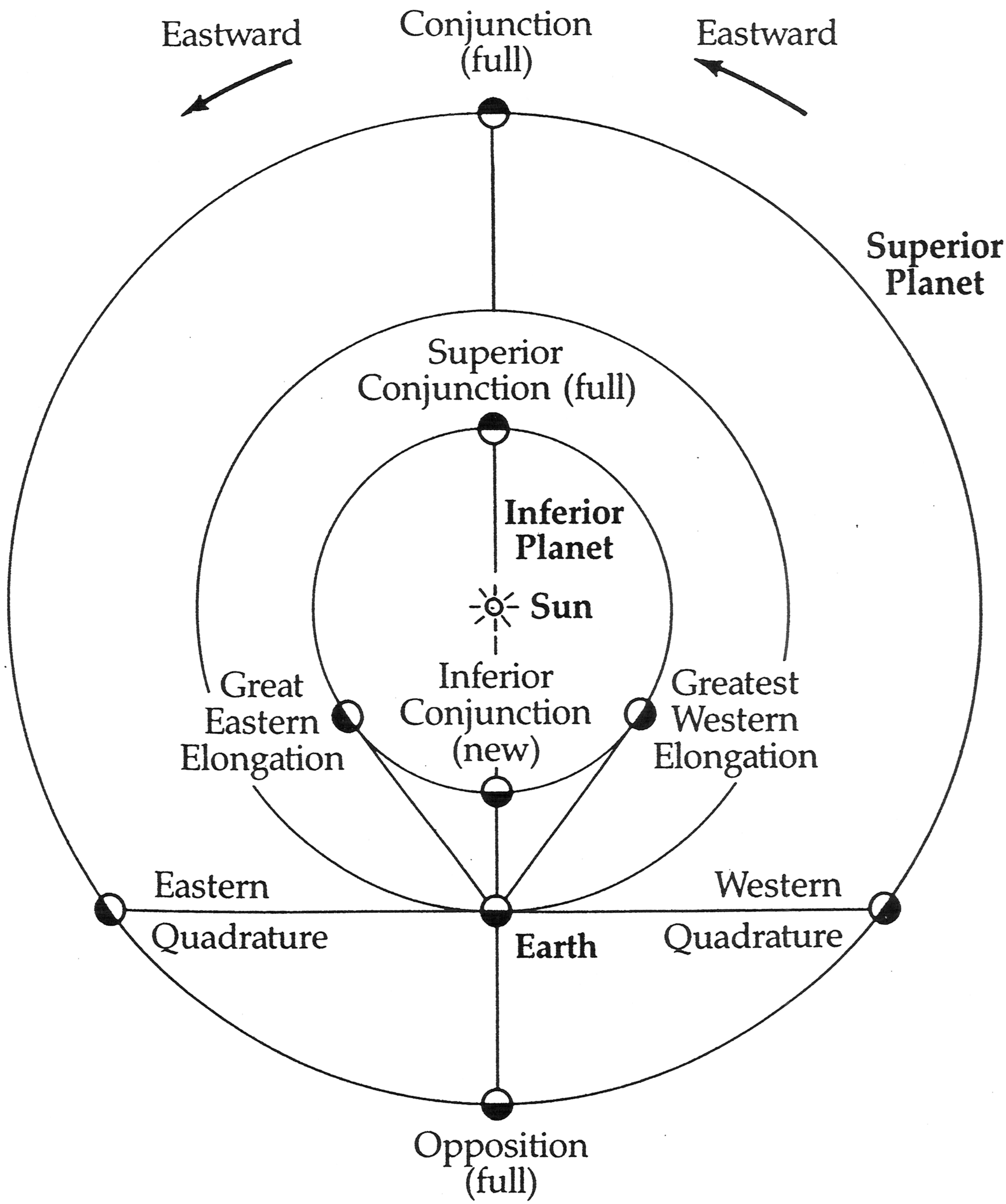

FIGURE 1–2 Heliocentric planetary configurations. Arrows indicate the direction of orbital motion as well as the rotational direction of the Earth. The phases of the planets' illumination are also shown.

3 Planetary Orbits

There are four key pieces of evidence for the revolution of the Earth about the sun (i.e. proving Copernicus was right), with just retrograde planetary motions dating from before the time of Kepler.

1. Retrograde motions: apparent reversals of direction in the paths of planets.

$\boxed{\textbf{Plot:}}$ inferior and superior planets and retrograde motions

2. Aberration of starlight: (James Bradley, 1729) the direction a telescope points toward a star varies by a small angle $\theta_{\rm aberr}$ on the celestial sphere, mapping out an approximate circle *over the year*. The aberration is due to the finite speed $v_\oplus$ in its revolution (plus the solar peculiar velocity component).

$$\theta_{\rm aberr} \;\approx\; \frac{v_\oplus}{c} \;\sim\; 20.49'' \tag{6}$$

so that $v_\oplus \sim 29.8$ km/sec. This **diurnal variation** (*over the year*) is most distinctive for stars well out of the ecliptic.

$*$ Note that the relevant angular scale is much larger than the sub-arcsecond ones encountered in studying stellar proper motions.

$\boxed{\textbf{Plot:}}$ Aberration of starlight

3. Stellar parallax: an additional *diurnal* variation is that nearby stars move slightly against background distant (and therefore fainter on average) stars.

$\boxed{\textbf{Plot:}}$ Parallactic orbits

Again, this effect is maximized for stars well out of the ecliptic, disappearing for those stars in the ecliptic plane.

$*$ Note that parallaxes are purely geometrical in space, so that the solar peculiar velocity is immaterial to their determination.

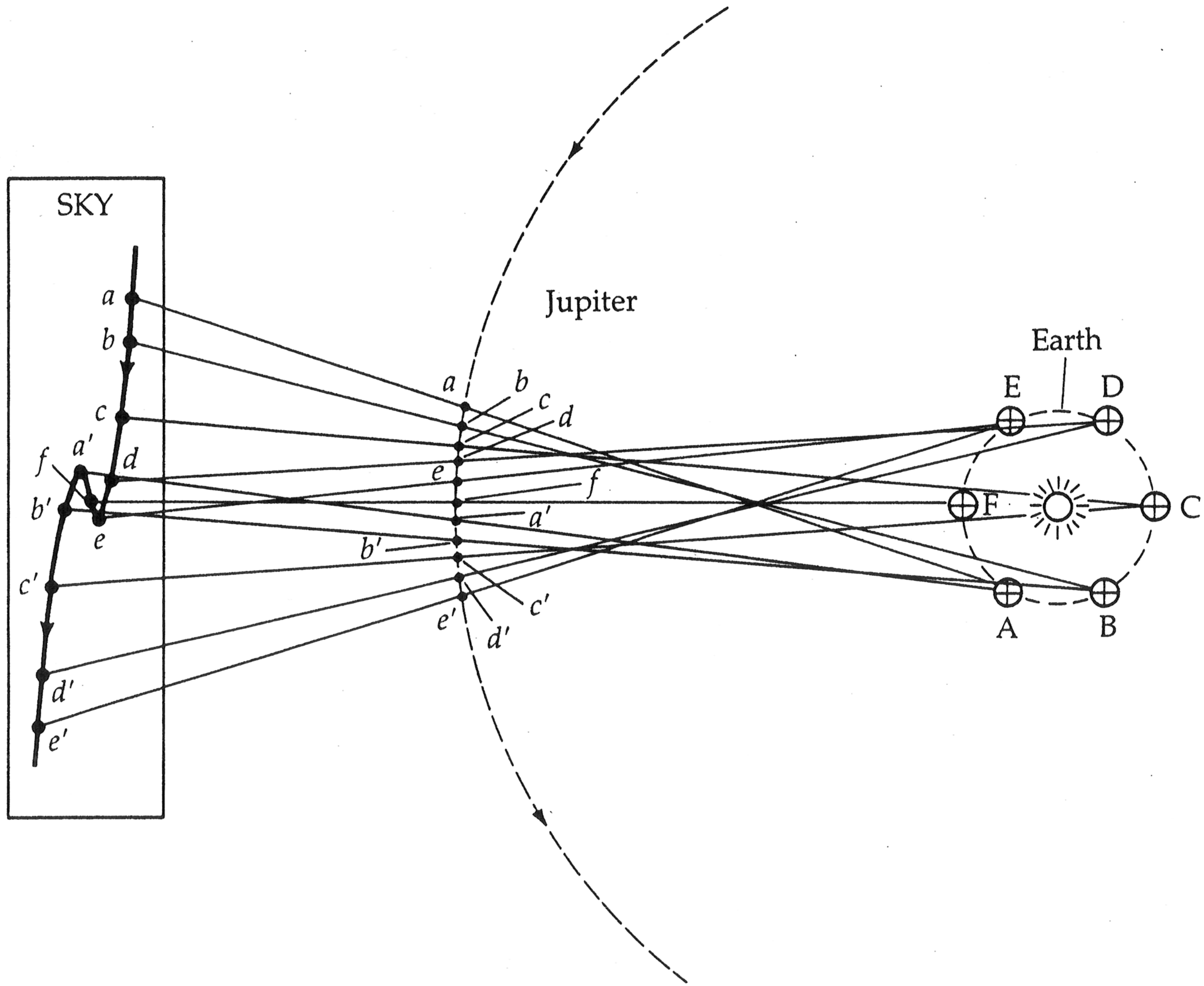

FIGURE 1–3 Retrograde motion in a heliocentric model. As the Earth passes a superior planet, that planet appears to move opposite its normal eastward direction with respect to the stars. Here the Earth passes Jupiter at point *Ff*, which marks the middle of the retrograde motion.

Handout:

Cartesian Form for Equation of an Ellipse

The ellipse possesses a semi-major axis of length a, a semi-minor axis of length b, and has an eccentricity e.

In general, $0 < e < 1$ for an ellipse, with $e = 0$ for a circle, and $e \lesssim 1$ for a cigar-shaped form.

- A formal definition of an ellipse is that the radial distances drawn to a point $P(x, y)$ on its periphery from the two foci sum to a constant:

$$r + r' \;=\; \text{constant} \;=\; 2a \quad . \tag{1}$$

In Cartesian coordinates, this can be written

$$\sqrt{(x+ae)^2 + y^2} \;+\; \sqrt{(x-ae)^2 + y^2} \;=\; 2a \quad , \tag{2}$$

which can be algebraically manipulated by completing the squares. To efficiently arrive at the familiar form, first multiply Eq. (2) by the difference of the two squares it contains. After slight rearrangement, this yields

$$\sqrt{(x+ae)^2 + y^2} \;-\; \sqrt{(x-ae)^2 + y^2} \;=\; 2ex \quad . \tag{3}$$

A sum and difference of Eqs. (2) and (3) then yields the compact symmetric forms

$$\sqrt{(x \pm ae)^2 + y^2} \;=\; a \pm ex \quad . \tag{4}$$

The product of these can then by substituted into the square of either Eq. (2) or Eq. (3) to derive the simple form

$$2\left\{x^2 + a^2e^2 + y^2\right\} + 2\left(a^2 - e^2x^2\right) \;=\; 4a^2 \quad , \tag{5}$$

which then trivially rearranges to

$$\frac{x^2}{a^2} + \frac{y^2}{a^2(1-e^2)} \;=\; 1 \quad . \tag{6}$$

At $x = 0$, $y = b$ and we deduce that $b^2 = a^2(1 - e^2)$. Hence

$$\boxed{\frac{x^2}{a^2} + \frac{y^2}{b^2} \;=\; 1} \tag{7}$$

is the familiar Cartesian form for the ellipse.