HOMEWORK PROBLEMS 8
ASTR 350

Compact Objects
Due date: Friday, 12/6/19 — 5pm

1. Carroll and Ostlie Problems 16.6 and 16.7.

   [14 points credit]

2. Carroll and Ostlie Problem 16.8.

   [14 points credit]

3. Carroll and Ostlie Problem 16.16.

   [14 points credit]

4. Carroll and Ostlie Problem 17.9.

   [14 points credit]

5. Carroll and Ostlie Problem 17.22. [Just for fun!]

   [14 points credit]

6. For young neutron stars like the Crab pulsar, the rotational energy loss or slowing (spin-down) is described by the differential equation with two contributions:

$$I\Omega\dot{\Omega} \; = \; -\alpha_{\rm EM}\Omega^4 - \alpha_{\rm GW}\Omega^6 \quad . \tag{1}$$

The left hand side uses a moment of inertia for the star I that is usually set to $10^{45}\,{\rm g\,cm}^2$. The first term on the left hand side is due to the electromagnetic dipole torque on the star with an inclination angle α between the magnetic and rotation axes. The second term is a gravitational quadrupole radiation term due to the oblateness ϵ of the star; it is a contribution from general relativity. The oblateness is the ratio of the difference in equatorial radii to the mean equatorial radius R. The coefficients are

$$\alpha_{\rm EM} \; = \; \frac{B_p^2 R^6 \sin^2\alpha}{6c^3} \quad , \quad \alpha_{\rm GW} \; = \; \frac{32 G I^2 \epsilon^2}{5c^5} \quad . \tag{2}$$

Here B_p is the surface polar field.

(a) Obtain an analytic solution to this ODE for the age $t(\Omega)$ of the Crab pulsar at the present angular frequency Ω in terms of the two α coefficients and the initial rotational angular speed Ω_i.

(b) If the measured period $P = 2\pi/\Omega$ and period derivative $\dot{P}$ for the Crab lead to a determination of the characteristic spin-down timescale of $\tau = -\Omega/\dot{\Omega} = 2513\,$years (in 2019), obtain the solution for the parameter $\alpha_{\rm GW}\Omega^2/\alpha_{\rm EM}$ at present, which defines the relative contributions of the gravitational and electromagnetic torques on the pulsar. Assume that the Crab exploded in 1054 AD and that now $\Omega \ll \Omega_i$.

(c) Hence obtain estimates for the values of $B_p \sin\alpha$ and ϵ. Also find the age of the Crab when the electromagnetic contribution to the spin-down dominates that due to gravitational radiation.

(d) What is the current ratio of the gravitational and electromagnetic energy loss rates, assuming that $\Omega_i = 10^4\,{\rm sec}^{-1}$, i.e the star is initially not far from the break-up rotational angular velocity.

[30 points credit]