Time limit: 50 minutes.
Instructions: For this test, you work in teams of eight to solve 15 short answer questions. All answers must be expressed in simplest form unless specified otherwise. Submit a single answer sheet for grading. Only answers written inside the boxes on the answer sheet will be considered for grading.
No calculators.

1. Two externally tangent unit circles are constructed inside square $ABCD$, one tangent to AB and AD, the other to BC and CD. Compute the length of AB.

2. We say that a triple of integers (a, b, c) is *sorted* if $a < b < c$. How many sorted triples of positive integers are there such that $c \leq 15$ and the greatest common divisor of a, b, and c is greater than 1?

3. Two players play a game where they alternate taking a positive integer N and decreasing it by some divisor n of N such that $n < N$. For example, if one player is given $N = 15$, they can choose $n = 3$ and give the other player $N - n = 15 - 3 = 12$. A player loses if they are given $N = 1$.

   For how many of the first 2015 positive integers is the player who moves first guaranteed to win, given optimal play from both players?

4. The polynomial $x^3 - 2015x^2 + mx + n$ has integer coefficients and has 3 distinct positive integer roots. One of the roots is the product of the two other roots. How many possible values are there for n?

5. You have a robot. Each morning the robot performs one of four actions, each with probability 1/4:

   - Nothing.
   - Self-destruct.
   - Create one clone.
   - Create two clones.

   Compute the probability that you eventually have no robots.

6. Four spheres of radius 1 are mutually tangent. What is the radius of the smallest sphere containing them?

7. Find the radius of the largest circle that lies above the x-axis and below the parabola $y = 2 - x^2$.

8. For some nonzero constant a, let $f(x) = e^{ax}$ and $g(x) = \frac{1}{a}\log x$. Find all possible values of a such that the graphs of f and g are tangent at exactly one point.

9. Consider a regular pentagon and connect each vertex to the pair of vertices farthest from it by line segments. The line segments intersect at 5 points to form another smaller pentagon. If the large pentagon has side length 1, compute the area of the smaller pentagon. Express your answer without trigonometric functions.

10. Let $f(x)$ be a function that satisfies $f(x)f(2-x) = x^2 f(x-2)$ and $f(1) = \frac{1}{403}$. Compute $f(2015)$.

11. You are playing a game on the number line. At the beginning of the game, every real number on $[0,4)$ is uncovered, and the rest are covered. A turn consists of picking a real number r such that, for all x where $r \le x < r+1$, x is uncovered. The turn ends by covering all such x. At the beginning of a turn, one selects such a real r uniformly at random from among all possible choices for r; the game ends when no such r exists. Compute the expected number of turns that will take place during this game.

12. Consider the recurrence:
$$a_{n+1} = 4a_n(1-a_n)$$
Call a point $a_0 \in [0,1]$ $q-$periodic if $a_q = a_0$. For example, $a_0 = 0$ is always a $q-$periodic fixed point for any q. Compute the number of positive $2015-$periodic fixed points.

13. Let $a, b, c \in \{-1, 1\}$. Evaluate the following expression, where the sum is taken over all possible choices of a, b, and c:
$$\sum abc(2^{\frac{1}{5}} + a2^{\frac{2}{5}} + b2^{\frac{3}{5}} + c2^{\frac{4}{5}})^4.$$

14. A small circle A of radius $\frac{1}{3}$ rotates, without slipping, inside and tangent to a unit circle B. Let p be a fixed point on A, and compute the length of the closed curve traced out by p as A rotates inside B.

15. Let x_1, x_2, x_3, x_4, x_5 be distinct positive integers such that $x_1+x_2+x_3+x_4+x_5 = 100$. Compute the maximum value of the expression
$$\frac{(x_2x_5+1)(x_3x_5+1)(x_4x_5+1)}{(x_2-x_1)(x_3-x_1)(x_4-x_1)} + \frac{(x_1x_5+1)(x_3x_5+1)(x_4x_5+1)}{(x_1-x_2)(x_3-x_2)(x_4-x_2)} + \frac{(x_1x_5+1)(x_2x_5+1)(x_4x_5+1)}{(x_1-x_3)(x_2-x_3)(x_4-x_3)} + \frac{(x_1x_5+1)(x_2x_5+1)(x_3x_5+1)}{(x_1-x_4)(x_2-x_4)(x_3-x_4)}.$$