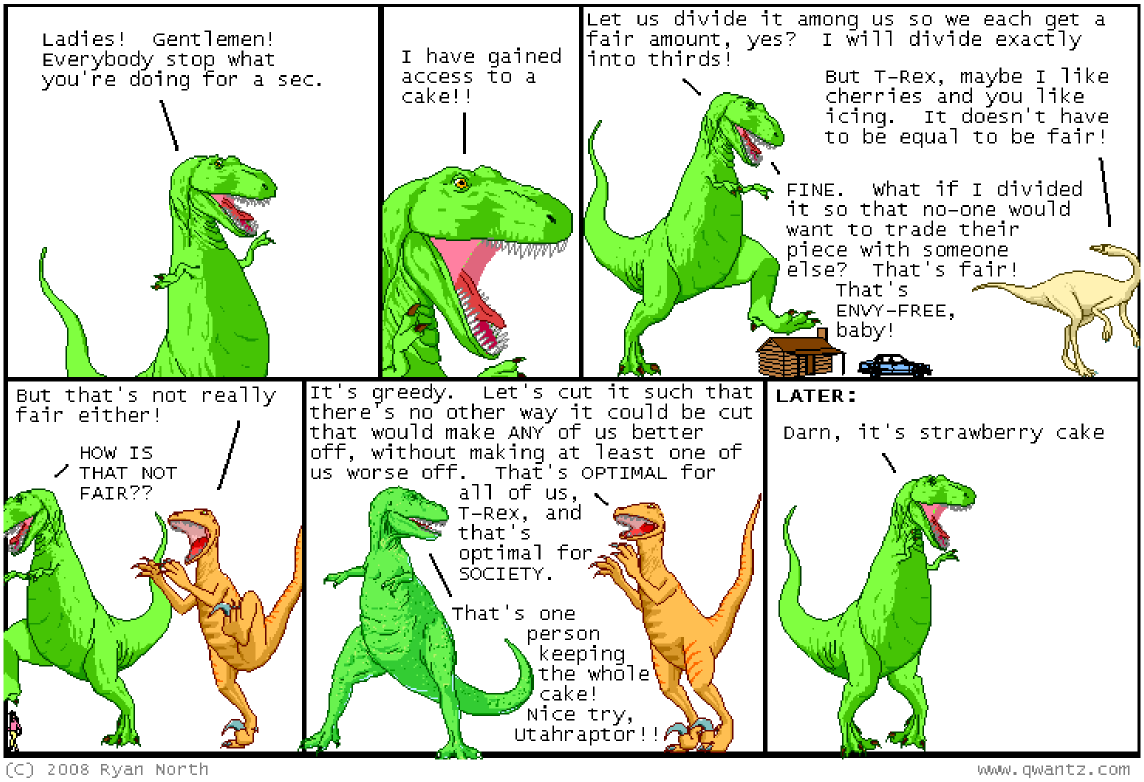

Introduction

Fair division is the process of dividing a set of goods among several people in a way that is fair'. However, as alluded to in the comic above, what exactly we mean by fairness is deceptively complex. We'll explore different notions of fairness in depth throughout this Power Round.

We'll begin with the canonical motivating example, called the *Cake-Cutting Problem*. Suppose you and your friend wish to split a cake. If the cake is homogeneous (all the same), then it is clearly most fair to split the cake in half, so you each receive an equal share. However, suppose half of the cake contains cherries, while the other half does not, and additionally suppose that you really like cherries, while your friend does not. In this case, splitting the cake equally is no longer obviously the best solution. Since you prefer cherry cake to regular cake while your friend does not, splitting the cake so you get the cherry half and your friend gets the rest is intuitively better than splitting both halves equally. These notions will be formalized shortly.

Definitions

All fair division problems share a few common features. First, they contain a set of n **players** which we will number Player 1 to Player n, and a finite set of **goods**, G, that we wish to divide among the players.

The elements of G may be one of three different types. Some goods, such as a car or a dog, are **indivisible**, which means they can only be assigned to a single player. Other goods, like money or cake are **divisible**, which means they can be divided among multiple players. We further distinguish between **homogeneous** divisible goods like money whose parts are indistinguishable from each other, and **heterogeneous** goods like cake, whose parts may not all be equivalent. In the example above, the half-cherry cake is heterogeneous and divisible, because it can be split among players, but its parts are distinguishable, since some will have more cherries than others.

Define a **division** to be an allocation of goods to players. We can denote divisions by listing the goods assigned to each player. For example, one possible division of the set of goods $G = \{A, B, C\}$ among two players is $(1 : (A, B), 2 : (C))$, which denotes that Player 1 receives A and B, while Player 2 receives C. If the goods are divisible and homogeneous, we can add a fraction to the good to denote the portion received by each player. For example, $(1 : (A, B, \frac{1}{3}C), 2 : (\frac{2}{3}C))$ denotes that Player 1 receives A, B, and one third of C, while Player 2 receives the remaining two thirds of C. Note that this notation isn't well-defined for heterogeneous goods, since if A is heterogeneous then "half of A" could mean any half, and they are not all necessarily equivalent.

Next, to model players' preferences, we assume each player has a **value function** that describes how much they value the goods in G. Define v_i to be player i's value function, and let $v_i(A)$ be the value player i places on the goods in set A ($A \subseteq G$). Value functions have the following properties:

i. Values placed on goods by players are always nonnegative real numbers.

ii. For each player i, $v_i(\{\}) = 0$ and $v_i(G) = 1$ (that is, all players place the same total value on all of G).

iii. Players' values are **additive**. That is, $v_i(A \cup B) = v_i(A) + v_i(B) - v_i(A \cap B)$ for any sets of goods A and B.

iv. Divisible, *homogeneous* goods are valued linearly. For example, if $v_i(\{g\}) = x$, and g is a homogeneous good, then player i would assign value $\frac{x}{2}$ to any half of g, $\frac{x}{5}$ to any fifth of g, etc. Note that this is *not* necessarily the case for heterogeneous goods, for example we've seen that not all cuts of a cake into halves will be valued equally by all players.

Hence, all players assign the same total value to all of G, and a player will never prefer *not* receiving a good g to receiving it. Additionally, the value function of any player can be enumerated by listing the value they place on each element of G. Note, however, that value functions can't necessarily be enumerated for all possible divisions of a heterogeneous good, since there may be uncountably many distinguishable divisions.

Define the **total value** that a player receives in a particular division to be the sum of values that player places on all goods they receive. For example, if $v_1(\{g_a\}) = 0.1$, $v_1(\{g_b\}) = 0.2$, and some division assigns goods g_a and g_b to Player 1, then the total value Player 1 receives from this division will be $v_1(\{g_a, g_b\}) = v_1(\{g_a\}) + v_1(\{g_b\}) = 0.1 + 0.2 = 0.3$. Note that total values depend on each player's *own* value functions, so the total value Player 2 receives from g_a and g_b is not necessarily the same as the total value Player 1 receives.

Finally, we assume that players are always **rational** and **honest**. This just means that players will always prefer to maximize the total value they receive, they will never prefer to receive less just to spite another player, and they will never lie about their true preferences until Problem 9.

Proportionality

A division is **proportional** if each of the n players receives a total value of at least $\frac{1}{n}$ according to *their own* value function.

A division is **super proportional** if each player receives total value *greater than* $\frac{1}{n}$ according to their own value function.

1. (a) [**6**] Suppose we wish to divide $G = \{A, B, C\}$, among two players (Player 1 and Player 2), and A, B, and C are indivisible. For each of the following, use the given value functions to find a division that satisfies proportionality. If a proportional division doesn't exist, write "no proportional division".

i.

	v_1	v_2
A	0.4	1.0
B	0.3	0
C	0.3	0

ii.

	v_1	v_2
A	0.6	0.6
B	0.4	0
C	0	0.4

iii.

	v_1	v_2
A	0.3	0.3
B	0.5	0.5
C	0.2	0.2

(b) [**4**] Suppose that all elements of G are divisible and homogeneous. Prove that a proportional division always exists.

(c) [**15**] Suppose that all elements of G are divisible and homogeneous. Prove that a super proportional division exists if and only if not all players have the same value function.

Solution to Problem 1:

(a) i. 1 gets $\{B, C\}$, 2 gets $\{A\}$.

ii. No proportional division exists. Both Player 1 and Player 2 must receive A in order to receive at least 0.5 total value; they cannot both receive A.

iii. 1 gets $\{A, C\}$, 2 get $\{B\}$.

(b) Give each player an equal proportion of all goods.

(c) First assume all value functions are the same. Then for any distribution of goods the total value received across all players must be 1. Thus all n players cannot receive more than $\frac{1}{n}$ total value from their share.

Next assume all players have different value functions. Begin with the proportional division which gives each player an equal proportion of all goods. Because all the value functions are different, each player must have another player for which they value some good less and some good more than the other player. If these two players swap, they are both better off and so receive more than $\frac{1}{n}$ total value. Once all players swap once, we have a super proportional division.

Envy-Free Divisions

A division is **envy-free** if every player believes, from their own perspective, that the total value they received is at least as high as any other players'. That is, if each player i is assigned goods G_i, then for each i and all $j \neq i$, $v_i(G_i) \geq v_i(G_j)$. In other words, no player is envious of any other.

For example, with two players and the value functions given in 1.a.i, the division $(1 : (B), 2 : (A, C))$ is not envy-free, because Player 1 received a total value of 0.3, while Player 1 perceives Player 2 to have received a total value of 0.7. Hence, Player 1 will be envious of Player 2.

2. (a) [**6**] Suppose we wish to divide the set $G = \{A, B, C, D\}$ among three players, and A, B, C, and D are indivisible. For each of the following, use the given value functions to find a division that is envy-free. If an envy-free division doesn't exist, write "no envy-free division".

i.

	v_1	v_2	v_3
A	0.25	0.3	0.5
B	0.25	0.4	0
C	0.25	0.2	0
D	0.25	0.1	0.5

ii.

	v_1	v_2	v_3
A	0.25	0.3	0.1
B	0.25	0.3	0.2
C	0.25	0.2	0.3
D	0.25	0.2	0.4

(b) [**4**] Prove that every envy-free division is also proportional.

(c) [**5**] With two players, is every proportional division also envy-free? Prove, or disprove by finding a counterexample.

(d) [**6**] With three players, is every proportional division also envy-free? Prove, or disprove by finding a counterexample.

Solution to Problem 2:

(a) i. Player 1 get $\{C, D\}$, Player 2 gets $\{B\}$, and player 3 gets $\{A\}$.

ii. There is no envy-free division. If players 1 or 2 get two objects, the other is envious unless they also get two objects. This would make the third person envious. Thus Player 1 and 2 must both get one item and so player 3 must get two items; however, in this case Player 1 is envious.

(b) If a division is envy-free, each player must receive at least $\frac{1}{n}$ value. Otherwise, the lower player would value the rest of the goods at larger than $\frac{n-1}{n}$, and so at least one of the $n-1$ other players receives more than $\frac{1}{n}$ from the the lowest player's perspective.

(c) In a proportional division, each player receives at least $\frac{1}{2}$ and perceive the value of the remaining goods at less than $\frac{1}{2}$.

(d) No, every proportional division is not envy-free:

	v_1	v_2	v_3
A	0.4	0.6	0
B	0	0.4	0.6
C	0.6	0	0.4

Give A to Player 1, B to Player 2, and C to player 3. This division is proportional but not envy free.

Efficiency

A division d_1 **dominates** another division d_2 if at least one player receives strictly greater total value in d_1 than d_2, and no player receives less total value in d_1 than d_2.

A division is **Pareto efficient** if it is not dominated by any other division. Equivalently, a division is Pareto efficient if all other divisions either assign identical total values to each player, or assign a lower total value to at least one player.

3. (a) [**6**] Suppose $G = \{A, B, C, D\}$ is a set of goods to be divided among two players, where A, B, C, and D are each indivisible. Given the following value functions for the players, list all Pareto efficient divisions, and indicate whether or not each one is proportional.

	v_1	v_2
A	0.3	0.1
B	0.3	0.2
C	0.2	0.3
D	0.2	0.4

(b) **[6]** Now, suppose $G = \{A, B, C\}$, A, B, and C are indivisible, and we wish to divide G among two players. Choose value functions for the two players, and find a proportional division that is not Pareto efficient.

(c) **[6]** Let G be a set of divisible, homogeneous goods. Prove, or disprove by finding a counterexample, that a division is *not* Pareto efficient if and only if there are goods g and g' in G such that a trade between two players of a part of g for a part of g' yields a division that dominates the given one.

Solution to Problem 3:

(a) i. 1 gets everything.
ii. 1 gets $\{A, B, C\}$, 2 gets $\{D\}$.
iii. 1 gets $\{A, B\}$, 2 gets $\{C, D\}$.
iv. 1 gets $\{A\}$, 2 gets $\{B, C, D\}$.
v. 2 gets everything.

(b) Player 1 gets $\{A, B\}$, Player 2 gets $\{C\}$. This is proportional but not Pareto efficient.

	v_1	v_2
A	0.5	0
B	0	0.5
C	0.5	0.5

(c) The statement is false.

	v_1	v_2	v_3
A	0.4	0.6	0
B	0	0.4	0.6
C	0.6	0	0.4

For a counter-example, consider the division where Player 1 receives $\{A\}$, Player 2 receives $\{B\}$, and player 3 receives $\{C\}$. This division is not Pareto efficient, but no two pairs of players want to trade a pair of goods.

Adjusted Winners

The **Adjusted Winner Procedure** guarantees a division for two players that is both envy-free and Pareto efficient when G contains divisible, homogeneous elements.

The adjusted winner procedure proceeds as follows:

i. Suppose $G = \{g_1, g_2, \ldots, g_k\}$ is a set of k divisible, homogeneous goods, and Players 1 and 2 have value functions v_1 and v_2, respectively.

ii. Let $G_1 = \{g_i \mid v_1(g_i) > v_2(g_i)\}$, let $G_2 = \{g_i \mid v_2(g_i) > v_1(g_i)\}$, and let G_r be the remainder, namely $\{g_i \mid v_1(g_i) = v_2(g_i)\}$. Without loss of generality suppose $v_1(G_1) \geq v_2(G_2)$.

iii. Tentatively assign the goods in $G_1 \cup G_r$ to Player 1, and the goods in G_2 to Player 2, so each player gets the goods they value higher than the other player, and Player 1 gets all goods for which the values were tied.

iv. List the goods on an order $g_{i_1}, g_{i_2}, \dots g_{i_k}$ such that

$$\frac{v_1(g_{i_1})}{v_2(g_{i_1})} \geq \frac{v_1(g_{i_2})}{v_2(g_{i_2})} \geq \dots \geq \frac{v_1(g_{i_k})}{v_2(g_{i_k})}$$

v. Then, Player 1 has been assigned goods $g_{i_1} \dots g_{i_r}$ for some r (in particular, all the goods with $\frac{v_1(g_{i_1})}{v_2(g_{i_1})} \geq 1$), and Player 2 has been assigned goods $g_{i_{r+1}} \dots g_{i_k}$.

vi. At this point, Player 1 has the equal or greater total value than Player 2. If their total values are equal, then we're done. Otherwise, Player 1 gives goods or fractions of goods to Player 2 in the order $g_{i_r}, g_{i_{r-1}}, g_{i_{r-2}}, \dots$ as necessary until both players have the same total value (based on their own value function).

In the following problem, we'll prove the fairness of the Adjusted Winner Procedure.

4. (a) **[4]** Suppose that Player 1 values g_i at least as much as Player 2 does, and Player 2 values g_j at least as much as Player 1 does. Additionally, suppose Player 1 possesses a fraction s $(0 < s \leq 1)$ of g_i, and Player 2 possesses a fraction t $(0 < t \leq 1)$ of g_j in some division. Prove that if a trade of Player 1's fraction of g_i for Player 2's fraction of g_j increases either player's total value, then the other player's total value must decrease.

   (b) **[6]** Suppose, as in (a), that Player 1 possesses a fraction s $(0 < s \leq 1)$ of g_i, and Player 2 possesses a fraction t $(0 < t \leq 1)$ of g_j in some division, and suppose $\frac{v_1(g_j)}{v_2(g_j)} \leq \frac{v_1(g_i)}{v_2(g_i)}$. Again, prove that if a trade of Player 1's fraction of g_i for Player 2's fraction of g_j increases either player's total value, then the other player's total value must decrease.

   (c) **[12]** Using previous results, prove that the Adjusted Winner Procedure terminates. Further, prove it yields a division that is both envy-free and Pareto efficient.

Solution to Problem 4:

(a) If Player 1 is happier after the trade:

$$t \cdot v_2(g_j) \geq t \cdot v_1(g_j) > s \cdot v_1(g_i) \geq s * v_2(g_i)$$

In this case, $t \cdot v_2(g_j) > s \cdot v_2(g_i)$ so Player 2 is unhappy with the trade.

(b) We can rearrange the provided equation as $\frac{v_1(g_j)}{v_1(g_i)} \leq \frac{v_2(g_j)}{v_2(g_i)}$. If Player 1 is happier after the trade, $t \cdot v_1(g_j) > s \cdot v_1(g_i)$. Thus $\frac{s}{t} < \frac{v_1(g_j)}{v_1(g_i)} \leq \frac{v_2(g_j)}{v_2(g_i)}$. Thus $s \cdot v_2(g_i) < t \cdot v_2(g_j)$, so Player 2 is unhappy with the trade.

(c) At the start of step (vi), Player 1 has equal or greater value than Player 2. If the algorithm transfers all goods to Player 2, Player 1 receives 0 value and Player 2 receives 1 value. By the intermediate value theorem, there must be an amount of goods transferred by the algorithm where Player 1 and Player 2 receive the same value. This procedure terminates at this middle point.
By part (b), the result of this procedure is Pareto efficient.
At the end of the algorithm, the values both players receive must be equal. If this value was below 0.5 for each, the distribution would not be Pareto efficient. Thus, the value must be above 0.5 for each so neither player envies the other's possessions.

Divide and Choose

One common solution to the cake-cutting problem described in the introduction and other two-player fair division problems is a procedure called **Divide and Choose**. In Divide and Choose, one player cuts the cake into any two pieces that they believe are of equal value. Then the second player chooses which of the pieces they prefer, and finally the first player receives the remaining piece.

More generally, Divide and Choose can be used for many fair division problems with two players. One player chooses a division of G into two sets, each of which would give him equal total value. Then, the other player chooses which set of goods she prefers, and the first player receives the remaining set. Note that the procedure doesn't always work if G contains an indivisible element, since in that case the first player can't necessarily divide G into two halves of equal value.

5. (a) **[8]** Let G be a cake with three divisible, homogeneous regions, C, V, and S (corresponding to chocolate, vanilla, and strawberry). Suppose Player 1 and Player 2 have the following value functions:

	v_1	v_2
C	0.2	0.3
V	0.3	0.4
S	0.5	0.3

If both players follow the procedure exactly, which of the following divisions could be the result of Divide and Choose? Note that either player could be the divider. For each case, write whether it is possible or not and which player was the divider.

i. $\{1 : (S),\ 2 : (C, V)\}$
ii. $\{1 : (C, V),\ 2 : (S)\}$
iii. $\{1 : (V, \frac{1}{2}S),\ 2 : (C, \frac{1}{2}S)\}$
iv. $\{1 : (\frac{1}{2}V, S),\ 2 : (C, \frac{1}{2}V)\}$

(b) **[4]** Prove, or disprove by finding a counterexample, that Divide and Choose always results in an envy-free division.

(c) **[6]** Prove, or disprove by finding a counterexample, that Divide and Choose always results in a Pareto efficient division.

(d) **[6]** Consider the following procedure, which attempts to generalize Divide and Choose to three players: Player 1 first creates a division of G of the goods into three sets, each of which would give him equal total value. Then, Player 2 chooses the set from the three that he prefers, Player 3 chooses the set from the remaining two that she most prefers, and finally Player 1 receives the last remaining set. Is this procedure envy-free? Prove, or disprove by finding a counterexample.

Solution to Problem 5:

(a) i. Yes; 1 was the divider.
ii. Not possible
iii. Not possible
iv. Yes; 2 was the divider.

(b) Both sets are equal so Player 1, so he will not be envious of the other player. Player 2 chose his higher value set, so he will not be envious of the other player.

(c) The statement is false.

	v_1	v_2
C	1	0
V	0	1

One possible execution of this process involves Player 1 dividing each item in half; in this case both players will receive 0.5 value. By taking their favorite item, both players could instead receive 1 value. Thus this execution of Divide and Choose is not Pareto efficient.

(d) Player 3 may still be envious of the set Player 2 receives.

Moving Knives

The special case where $G = \{g\}$ contains a single heterogeneous element (for example, if g is a cake) is common, and a number of fair division procedures have been developed specifically for it. One class of such procedures is called the **Moving Knife Procedures**.

For this problem, we make a few simplifying assumptions. First, let the real numbers in the interval $(0, 1)$ correspond to horizontal position across the width of g, which we imagine is similar to a cake. Restrict the cuts we can make in the cake to cuts at positions $x \in (0, 1)$. Then we can describe a division of $G = \{g\}$ by the numeric positions of the cuts we make on g, and the allocation of the resulting pieces to players. Additionally, this restriction allows us to model each player's value function as a nonnegative function on the interval $[0, 1]$ such that if f is a value function then $\int_0^1 f(x)\,dx = 1$ and $\int_a^b f(x)\,dx$ is the value the player will place on the piece between cuts at a and b. Finally, we assume that these value functions are continuous, so these integrals are well-behaved.

6. (a) **[4]** The two-player Moving Knife Procedure proceeds as follows: A "knife" is held over g at position 0, and slowly swept across so the numeric position of the knife gradually increases. At any point, if either player believes that the portion of g to the left of the knife has half the value of all of g, they call a stop to the procedure, and g is cut at that point. Then, the player who stopped the knife receives the left half and the other player receives the right half. Prove that, like Divide and Choose, this procedure is results in an envy-free division.

   (b) **[8]** A nice advantage of this procedure is that it easily generalizes to more than two people. Consider the following modification for n people: The knife is swept as before, but as soon as any of the n players believe the portion of g to the left of the knife has value $\frac{1}{n}$ of the whole, they call stop. Then, g is cut at that point, and the player who called stop receives the piece to the left of the knife. Then, the entire procedure is repeated with the remainder of g and the remaining $n - 1$ players. Prove that this procedure results in a proportional division.

   (c) **[8]** Consider the following variant of the two-player moving knife procedure: Player 1 holds two knives. The first is initially at the left edge of g, and the second is placed at the line that Player 1 believes splits g into two halves of equal value. Then, Player 1 sweeps both knives slowly to the right, such the portion of g between the knives remains exactly half the value of all of g. As soon as Player 2 agrees that the portion between the knives is half the value of all of g, she tells Player 1 to stop. Then, g is cut at the position of the knives, the center piece is given to Player 2, and the remainder is given to Player 1.

      i. Prove that this procedure always terminates.

      ii. Prove that this procedure results in a division where both players believe that they and the other player received a total value of exactly $\frac{1}{2}$. Such a division is called an **exact** division, and is clearly also envy-free and proportional.

Solution to Problem 6:

(a) This is essentially the "Divide and Choose" procedure.

(b) We will use induction on the number of players. The base case is part (a). At the beginning of the procedure, one of the players stops the sweep when he believes the portion of g to the left of the knife is $\frac{1}{n}$. At this point, all other players believe at least

$\frac{n-1}{n}$ of the value to be on the other side of the cake. By the inductive hypothesis each of the remaining players will also receive a piece worth at least $\frac{1}{n-1} * \frac{n-1}{n} = \frac{1}{n}$.

(c) Let $f(t)$ be the value for Player 1 of the center slice at time t. Let $g(t)$ be the value for Player 2 of the center slice at time t. Note $f(t) = \frac{1}{2}$. If $g(0) = \frac{1}{2}$, then the procedure immediately terminates. If $g(0) > \frac{1}{2}$, then at some later point $g(t) < \frac{1}{2}$. If $g(0) < \frac{1}{2}$, then at some later point $g(t) > \frac{1}{2}$. In either case, by the Intermediate Value Theorem there is some middle point t' where $g(t') = \frac{1}{2}$. The algorithm will terminate here, and both players will believe both players got exactly $\frac{1}{2}$.

Lying for Fun and Profit

To this point, we've assumed that the players are always honest, and accurately represent their true preferences. However, what happens when we remove this assumption?

7. (a) [**6**] Let $G = \{A, B, C\}$, where A, B, and C are each divisible and homogeneous. Additionally, suppose Player 1 and Player 2's true value functions are as follows:

	v_1	v_2
A	0.2	0.3
B	0.3	0.4
C	0.5	0.3

If Player 1 and Player 2 divide G using the Divide and Choose method, and Player 1 knows Player 2's value function (as well as his own), then Player 1 can guarantee himself a total value of $\frac{x}{1000}$ for many integers x by carefully selecting the initial division (knowing that Player 2 will then choose whichever half has greatest total value to her). Compute the largest such integer x.

(b) [**4**] Suppose a division is chosen by a third party that maximizes the total value received by both players. For example, using the values given in part (a), the maximal division is $\{1 : (C), 2 : (A, B)\}$, which yields a total value of 0.5 to Player 1. If Player 1 lied about his value function to the third party but Player 2 told the truth, what is the maximal true total value (using his original value function from (a)) that Player 1 could achieve?

Solution to Problem 7:

(a) Player 1 wants to get as much of B and C as possible, prioritizing C above B. To achieve this, for any $\epsilon > 0$ Player 1 can split the bundle as:

$$\{\{C, (\frac{1}{2} - \epsilon)B\}, \{(\frac{1}{2} + \epsilon)B, A\}\}$$

This ensures that Player 2 will take the second bundle to get slightly over $\frac{1}{2}$ value. Thus Player 1 can get $0.5 + (\frac{1}{2} + \epsilon)0.3 = \frac{650}{1000} - \frac{1}{2}\epsilon$ value. Therefore the highest possible integer x is 649.

(b) To acquire all of B and C, Player 1 can set his value for B and C slightly higher than Player 2's value for B and C. This ensures that Player 1 will obtain all of B and C for a total value of 0.8.