1. Clyde is making a Pacman sticker to put on his laptop. A Pacman sticker is a circular sticker of radius 3 inches with a sector of 120° cut out. What is the perimeter of the Pacman sticker in inches?

   Answer: $\mathbf{4\pi + 6}$

   Solution: The perimeter of a circle with radius 3 in is $2\pi r = 6\pi$. The sector cut out decreases the perimeter by $\frac{120}{360} = \frac{1}{3}$ of its perimeter and adds in two lines of length 3. Thus, the perimeter of the sticker is $\frac{2}{3}(6\pi) + 2 \cdot 3 = \boxed{4\pi + 6}$.

2. In a certain right triangle, dropping an altitude to the hypotenuse divides the hypotenuse into two segments of length 2 and 3 respectively. What is the area of the triangle?

   Answer: $\mathbf{\frac{5\sqrt{6}}{2}}$

   Solution: Denote the right triangle ABC with hypotenuse BC. Let D be the intersection of the altitude and BC and let $CD = 2$ and $BD = 3$. Triangle ACD is similar to triangle ABC so $\frac{AC}{CD} = \frac{BC}{AC}$. Thus, $AC = \sqrt{BC \cdot CD} = \sqrt{5 \cdot 2} = \sqrt{10}$. Triangle ABD is similar to triangle ABC so $\frac{AB}{BD} = \frac{BC}{AB}$. Thus, $AB = \sqrt{BC \cdot BD} = \sqrt{5 \cdot 3} = \sqrt{15}$. Therefore, the area of ABC is $\frac{1}{2} \cdot \sqrt{10} \cdot \sqrt{15} = \boxed{\frac{5\sqrt{6}}{2}}$.

3. Consider a triangular pyramid $ABCD$ with equilateral base ABC of side length 1. $AD = BD = CD$ and $\angle ADB = \angle BDC = \angle ADC = 90°$. Find the volume of $ABCD$.

   Answer: $\mathbf{\frac{\sqrt{2}}{24}}$

   Solution: Let E be the center of equilateral triangle ABC so that DE is the height of the pyramid. Then AE is the distance from a vertices of equilateral triangle ABC to its centroid, and so is $\frac{2}{3}\frac{sqrt3}{2} = \frac{1}{\sqrt{3}}$. Since $AD = BD$ and $\angle ADB = 90°$, ADB is a 45-45-90 triangle and hence $AD = \frac{AB}{\sqrt{2}} = \frac{1}{\sqrt{2}}$. Thus, by Pythagoras, $DE = \sqrt{AD^2 - AE^2} = \frac{1}{\sqrt{6}}$. Now, the area of the base ABC is $\frac{\sqrt{3}}{4}$ so the volume of $ABCD$ is $\frac{1}{3} \cdot \frac{1}{\sqrt{6}} \cdot \frac{\sqrt{3}}{4} = \boxed{\frac{\sqrt{2}}{24}}$.

4. Two circles with centers A and B respectively intersect at two points C and D. Given that A, B, C, D lie on a circle of radius 3 and circle A has radius 2, what is the radius of circle B?

   Answer: $\mathbf{4\sqrt{2}}$

   Solution: First, note that by symmetry, $\angle ACB = \angle ADB$. Next, since A, B, C, D lie on a circle, the quadrilateral $ACBD$ is cyclic and hence opposite corners $\angle ACB$ and $\angle ADB$ sum to 180°. Therefore, it follows that $\angle ACB = \angle ADB = 90°$ so AB must be the diameter of the circle containing points A, B, C, D. Since this circle has radius 3, $AB = 6$. Next, AC is a radius of circle A so $AC = 2$ and BC is a radius of circle B. Applying Pythagoras to the triangle ABC, we have

$$\begin{aligned} AC^2 + BC^2 &= AB^2 \\ 2^2 + BC^2 &= 6^2 \\ BC^2 &= 32 \\ BC &= \boxed{4\sqrt{2}} \end{aligned}$$

5. Consider two concentric circles of radius 1 and 2. Up to rotation, there are two distinct equilateral triangles with two vertices on the circle of radius 2 and the remaining vertex on the circle of radius 1. The larger of these triangles has sides of length a, and the smaller has sides of length b. Compute $a + b$.

   Answer: $\sqrt{15}$

   Solution 1: Let a equilateral triangle ABC have A lie on the circle of radius 1 and B, C lie on the circle of radius 2. Since ABC is equilateral and BC is a chord of the circle of radius 2, the center of the circles and A must lie on the perpendicular bisector of BC. We see that the two configurations correspond to where A, B, C all lie on the same semicircle and where A, B, C do not all lie on the same semicircle.

   We first solve for the side length when A, B, C do not all lie on the same semicircle. Let O denote the center of circle and let D denote the midpoint of BC. In addition, let s denote the side length of ABC. Since A, B, C do not all lie on the same semicircle, we must have O inside A.

   Since ABC is equilateral, it must have height $AD = \frac{\sqrt{3}s}{2}$. In addition, we know that $BD = \frac{s}{2}$, $AO = 1$, and $BO = 2$. Thus, $DO = AD - AO = \frac{\sqrt{3}s-2}{2}$. Now, applying the Pythagorean theorem to triangle BDO, we have

$$BD^2 + DO^2 = BO^2$$
$$\left(\frac{s}{2}\right)^2 + \left(\frac{\sqrt{3}s - 2}{2}\right)^2 = 2^2$$
$$s^2 + 3s^2 - 4\sqrt{3}s + 4 = 16$$
$$s^2 - \sqrt{3}s - 3 = 0$$

   Thus, it follows that $s = \frac{\sqrt{3}\pm\sqrt{3+4\cdot 3}}{2}$. The side length of the equilateral triangle is thus the positive value $s = \frac{\sqrt{3}+\sqrt{15}}{2}$.

   Next, suppose A, B, C all lie on the same semicircle. Then O does not lie inside ABC. Again, let s denote the side length of ABC. We still have $AD = \frac{\sqrt{3}s}{2}$, $AO = 1$, $BO = 2$, $BD = \frac{s}{2}$, but this time $DO = AD + AO = \frac{\sqrt{3}s+2}{2}$. Applying the Pythagorean theorem to triangle BDO again, we have

$$BD^2 + DO^2 = BO^2$$
$$\left(\frac{s}{2}\right)^2 + \left(\frac{\sqrt{3}s + 2}{2}\right)^2 = 2^2$$
$$s^2 + 3s^2 + 4\sqrt{3}s + 4 = 16$$
$$s^2 + \sqrt{3}s - 3 = 0$$

   So $s = \frac{-\sqrt{3}+\sqrt{15}}{2}$. The sum of the two possible side lengths is therefore $\frac{\sqrt{3}+\sqrt{15}}{2} + \frac{-\sqrt{3}+\sqrt{15}}{2} = \boxed{\sqrt{15}}$.

   Solution 2: Let the smaller triangle be ABC and the larger triangle be $A'B'C'$. Let the center of the circles with O, and without loss of generality, let A and A' be coincident. Finally, let B and B' be on opposite sides of the line AO. Then by symmetry we have that lines BB' and CC'

form a pair of intersecting chords in the circle of radius 2, intersecting at $A = A'$. Let the side length of ABC be a and the side length of $A'B'C'$ be b. Draw the diameter $\overline{OA}$, intersecting the radius 2 circle at points X and Y, and use power of a point to see that the power of $A = A'$ is $(AX)(AY) = 3 \cdot 1 = 3$. Thus, $(AB)(A'B') = (AC)(A'C') = ab = 3$.

Now consider the point E where $A'B'$ intersects the circle of radius 1. Drop a perpendicular from O to the point D on $\overline{A'B'}$. The triangle $OA'D$ is then a 30-60-90 triangle with hypotenuse of length 1. Thus, $A'D = \sqrt{3}/2$, and $A'E = \sqrt{3}$, as AOE is isosceles. Finally, note that by symmetry, $BA' = EB' = a$. But since $A'B' = b = AE + EB' = \sqrt{3} + a$, we have that $b = \sqrt{3} + a$.

Plugging this in to $ab = 3$, we solve for a and b and find that $a + b = \boxed{\sqrt{15}}$

6. In a triangle ABC, let D and E trisect BC, so $BD = DE = EC$. Let F be the point on AB such that $\dfrac{AF}{FB} = 2$, and G on AC such that $\dfrac{AG}{GC} = \dfrac{1}{2}$. Let P be the intersection of DG and EF, and extend AP to intersect BC at a point X. Find $\dfrac{BX}{XC}$.

   Answer: $\mathbf{\dfrac{2}{3}}$

   Solution: Note that DG happens to be parallel to AB as $\frac{BD}{DC} = \frac{AG}{GC} = \frac{1}{2}$. Therefore triangles DEP and BEF are similar so we have $\frac{DP}{BF} = \frac{DE}{BE} = \frac{1}{2}$. This implies that $DP = \frac{BF}{2} = \frac{AB}{6}$. Next, triangles DPX and ABX are similar so we have $\frac{BX}{DX} = \frac{AB}{PD} = 6$. Hence, $BX = \frac{6}{5}BD = \frac{2}{5}BC$ and $XC = BC - BX = \frac{3}{5}BC$. So we conclude that $\frac{BX}{XC} = \boxed{\dfrac{2}{3}}$.

7. A unit sphere is centered at $(0, 0, 1)$. There is a point light source located at $(1, 0, 4)$ that sends out light uniformly in every direction but is blocked by the sphere. What is the area of the sphere's shadow on the x-y plane? (A point (a, b, c) denotes the point in three dimensions with x-coordinate a, y-coordinate b, and z-coordinate c).

   Answer: $\mathbf{\frac{3\sqrt{2}\pi}{2}}$

   Solution: The region in space that is in shadow due to the sphere is a cone. Therefore, the sphere's shadow on the xy plane is the intersection of a cone and a plane, which is an ellipse. We proceed to compute the major and minor axes of the ellipse.

   First, note that since the y-coordinate of the sphere's center and the light source both equal 0, one of the axes must lie along the x-axis. The axes of an ellipse are perpendicular to one another, so the remaining axis must be parallel to the y-axis.

   Now, consider projecting everything onto the xz plane (that is, simply disregard the y coordinate). The sphere is projected onto a unit circle centerd at $(0, 1)$, the light source is projected to the point $(1, 4)$, and the ellipse is projected onto its horizontal axis. Let ABC be the triangle consisting of the light source A and let B, C be the two ends of the ellipse's axis. The circle is thus the incircle of ABC, and we see that ABC must be a right angle triangle with $\angle ABC = 90°$. Let D be the point where the incircle intersects AB, E be the point where the incircle intersects BC, and F be the point where the incircle intersects AC. Then $AD = AF = 3$, $BD = BE = 1$ and $CF = CE$. By Pythagoras, $AB^2 + BC^2 = AC^2$ so $4^2 + (1 + CE)^2 = (3 + CE)^2$. Solving for CE, we find $CE = 2$, so the horizontal axis of the ellipse $BC = 3$.

   Next, we project everything onto the yz plane. This time, the ellipse is projected onto its vertical axis. Again, let A be the light source and B, C be the endpoints of the ellipse's axis. Then ABC is a isoceles triangle with $AB = BC$ and the unit sphere is projected onto the incircle of ABC.

If we let D be the intersection of the incircle and AB, E be the intersection of the incircle and AC, and F be the intersection of the incircle and BC, then we have $CE = CF = BD = BF$ and $AD = AE$. Let O denote the center of the incircle. Then $OA = 3$ and $OD = OE = OF = 1$. By Pythagoras, $AE^2 + OD^2 = OA^2$ so $AE = \sqrt{3^2 - 1^2} = 2\sqrt{2}$. Applying Pythagoras again, to ACF, we have $AF^2 + CF^2 = AC^2$ so $4^2 + CF^2 = (2\sqrt{2} + CF)^2$. Solving for CF, we have $CF = \sqrt{2}$. Thus, the vertical axis BC is equal to $2 \cdot CF = 2\sqrt{2}$.

The sphere's shadow on the xy plane is hence an ellipse with axes 3 and $2\sqrt{2}$ so the area of the shadow is $\frac{3}{2} \cdot \frac{2\sqrt{2}}{2} \cdot \pi = \boxed{\frac{3\sqrt{2}\pi}{2}}$.

8. Consider the parallelogram $ABCD$ such that $CD = 8$ and $BC = 14$. The diagonals $\overline{AC}$ and $\overline{BD}$ intersect at E and $AC = 16$. Consider a point F on the segment $\overline{ED}$ with $FD = \frac{\sqrt{66}}{3}$. Compute CF.

Answer: $\sqrt{\frac{148}{3}}$

Solution 1: First, note that in a parallelogram the diagonals bisect each other so $AE = CE = \frac{AC}{2} = 8$ and $BE = DE$. Thus, triangle CDE is isoceles with $CD = CE = 8$. Drop an altitude CG from C onto DE. Then $DG = EG$ and $BG = 3EG$. Applying Pythagoras to triangles CEG and CBG, we have $CE^2 - EG^2 = CG^2 = CB^2 - BG^2$. Thus,

$$\begin{aligned} 8^2 - EG^2 &= 14^2 - (3EG)^2 \\ 8EG^2 &= 132 \\ EG &= \frac{\sqrt{66}}{2} \end{aligned}$$

and the altitude is $CG = \sqrt{CE^2 - EG^2} = \sqrt{64 - \frac{66}{4}} = \frac{\sqrt{190}}{2}$. Now, since $FG = DG - FD = EG - FD = \frac{\sqrt{66}}{2} - \frac{\sqrt{66}}{3} = \frac{\sqrt{66}}{6}$. Applying Pythagoras to triangle CFG, we have

$$\begin{aligned} CF^2 &= FG^2 + CG^2 \\ &= \frac{66}{36} + \frac{190}{4} \\ &= \frac{148}{3} \end{aligned}$$

so $CF = \boxed{\sqrt{\frac{148}{3}}}$.

Solution 2: By the parallelogram law,

$$\begin{aligned} (AD)^2 + (BC)^2 + (AB)^2 + (CD)^2 &= (AC)^2 + (BD)^2 \\ 14^2 + 14^2 + 8^2 + 8^2 &= 16^2 + (BD)^2 \\ (BD)^2 &= 264 \\ BD &= 2\sqrt{66} \end{aligned}$$

Thus

$$EF = \frac{2\sqrt{66}}{3}$$

Let $x = CF$.

By Stewart's Theorem:

$$\begin{aligned}
8 \cdot \frac{\sqrt{66}}{3} \cdot 8 + 8 \cdot \frac{2\sqrt{66}}{3} \cdot 8 &= x \cdot \sqrt{66} \cdot x + \sqrt{66} \cdot \frac{2\sqrt{66}}{3} \cdot \frac{\sqrt{66}}{3} \\
\frac{64\sqrt{66}}{3} + \frac{128\sqrt{66}}{3} &= x^2\sqrt{66} + \frac{132\sqrt{66}}{9} \\
64\sqrt{66} &= x^2\sqrt{66} + \frac{44\sqrt{66}}{3} \\
64 &= x^2 + \frac{44}{3} \\
x^2 &= \frac{192 - 44}{3} \\
x &= \sqrt{\frac{148}{3}}
\end{aligned}$$

9. Triangle ABC is isoceles with $AB = AC = 2$ and $BC = 1$. Point D lies on AB such that the inradius of ADC equals the inradius of BDC. What is the inradius of ADC?

Answer: $\frac{\sqrt{15}-\sqrt{3}}{8}$

Solution: Now, let y denote CD and let x denote BD so $AD = 2 - x$. Since the area of a triangle is equal to its semiperimeter times its inradius and triangle ADC and BDC have the same inradius, the ratio of their areas is the ratio of their semiperimeters. Thus, $\frac{\Delta ADC}{\Delta BDC} = \frac{4-x+y}{1+y+x}$. However, the ratio of their areas is also equal to the ratio $\frac{AD}{BD}$. Thus, we have that

$$\begin{aligned}
\frac{4 - x + y}{1 + y + x} &= \frac{2 - x}{x} \\
4x - x^2 + xy &= 2 + 2x + 2y - x - x^2 - xy \\
2y - 2xy &= 3x - 2 \\
y &= \frac{3x - 2}{2 - 2x}
\end{aligned}$$

Next, note that $\cos(\angle ABC) = \frac{1}{4}$ since the altitude from A bisects BC. Applying the law of cosines to triangle BDC, we have

$$\begin{aligned}
y^2 &= x^2 + 1^2 - 2x\cos(\angle ABC) \\
&= x^2 - \frac{x}{2} + 1
\end{aligned}$$

Combining these two equations, we can solve for x:

$$\begin{aligned}
x^2 - \frac{x}{2} + 1 &= \left(\frac{3x - 2}{2 - 2x}\right)^2 \\
(2 - 2x)^2(2x^2 - x + 2) &= 2(3x - 2)^2 \\
8x^4 - 20x^3 + 24x^2 - 20x + 8 &= 18x^2 - 24x + 8 \\
8x^4 - 20x^3 + 6x^2 + 4x &= 0 \\
4x^4 - 10x^3 + 3x^2 + 2x &= 0
\end{aligned}$$

Now, notice that $x = 0$ and $x = 2$ are extraneous solutions so we may divide out by x and $(x-2)$ to obtain the quadratic $4x^2 - 2x - 1$ which has solutions $x = \frac{1\pm\sqrt{5}}{4}$. One solution is negative so we may discard it and hence we conclude that $x = \frac{1+\sqrt{5}}{4}$. Plugging in x into the equation $y = \frac{3x-2}{2-2x}$, we see that $y = \frac{\sqrt{5}}{2}$.

Now, let r denote the inradius of ADC, which is equal to the inradius of BDC. We have that $\Delta ADC + \Delta BDC = \Delta ABC$. The height of triangle ABC is $\sqrt{2^2 - \left(\frac{1}{2}\right)^2} = \frac{\sqrt{15}}{2}$ so the area of ABC is $\frac{1}{2} \cdot \frac{\sqrt{15}}{2} \cdot 1 = \frac{\sqrt{15}}{4}$. The area of ADC is its semiperimeter $\frac{4-x+y}{2}$ times r and the area of BDC is its semiperimeter $\frac{1+x+y}{2}$ times r. Thus, we have that

$$\begin{aligned}
\frac{4-x+y}{2}r + \frac{1+x+y}{2}r &= \frac{\sqrt{15}}{4} \\
(5+2y)r &= \frac{\sqrt{15}}{2} \\
(5+\sqrt{5})r &= \frac{\sqrt{15}}{2} \\
r &= \frac{\sqrt{15}}{2(5+\sqrt{5})} \\
r &= \boxed{\frac{\sqrt{15}-\sqrt{3}}{8}}
\end{aligned}$$

10. For a positive real number k and an even integer $n \geq 4$, the k-*Perfect* n-gon is defined to be the equiangular n-gon $P_1P_2\dots P_n$ with $P_iP_{i+1} = P_{n/2+i}P_{n/2+i+1} = k^{i-1}$ for all $i \in \{1, 2, \dots, n/2\}$, assuming the convention $P_{n+1} = P_1$ (i.e. the numbering wraps around). If $a(k, n)$ denotes the area of the k-*Perfect* n-gon, compute $\frac{a(2,24)}{a(4,12)}$.

Answer: $\mathbf{5 - \frac{25}{4}\sqrt{2} + \frac{25}{4}\sqrt{6}}$

Solution 1: We find a general formula for $\dfrac{a(k, 4n)}{a(k^2, 2n)}$.

Let $P_1P_2\dots P_{4n}$ be the k-*Perfect* $4n$-gon. Consider the $2n$-gon $P_1P_3\dots P_{4n-1}$, obtained by taking every other vertex starting with P_1.

For any i, $1 \leq i \leq 2n-2$, $\triangle P_1P_2P_3 \sim \triangle P_iP_{i+1}P_{i+2}$ with a ratio of $k^{i-1} : 1$, by SAS similarity. Therefore, $P_iP_{i+2} = k^iP_1P_3$ for any such i. Similarly, for i with $2n \leq i \leq 4n-2$, we have $P_iP_{i+2} = k^{i-2n+1}P_1P_3$. So, we conclude that $P_1P_3\dots P_{4n-1}$ is similar to the k^2-*Perfect* $2n$-gon, by a ratio of $P_1P_3 : 1$.

By the Law of Cosines,

$$P_1P_3 = \sqrt{1^2 + k^2 - 2 \cdot 1 \cdot k\cos\left(\pi - \frac{2\pi}{4n}\right)} = \sqrt{1 + k^2 + 2k\cos\left(\frac{\pi}{2n}\right)}.$$

Therefore, the area of $P_1P_3\dots P_{4n-1}$ is

$$\left(1 + k^2 + 2k\cos\left(\frac{\pi}{2n}\right)\right)a(k^2, 2n).$$

If we remove this $2n$-gon from our larger $4n$-gon, we are left with $2n$ similar triangles. Each has an angle of $\pi - \frac{\pi}{2n}$ with incident edges in a ratio of $1 : k$. For each $i \in \{0, 2, \dots, n-1\}$, there are two such triangles where the edges incident have lengths k^{2i} and k^{2i+1}. We want to relate the sum of the areas of these triangles to $a(k^2, 2n)$ somehow.

Let A be a point in the plane, and construct rays $\overrightarrow{AB_0}, \overrightarrow{AB_1}, \overrightarrow{AB_2}, \dots, \overrightarrow{AB_n}$ all emanating from A such that $\overrightarrow{AB_i}$ is $\frac{\pi}{n}$ radians clockwise with respect to $\overrightarrow{AB_{i-1}}$. Note that this makes points B_0, B_n, and A collinear. Now, for each $i \in \{0, \dots, n\}$, let C_i be the point on $\overrightarrow{AB_i}$ that is k^{2i} units from A. Consider the $n+1$-gon $C_0C_1 \dots C_n$. For each $i \in \{0, \dots, n-1\}$, $\triangle AC_0C_1 \sim \triangle AC_iC_{i+1}$ with ratio k^{2i}. This implies that $C_{i-1}C_i = k^{2i}C_0C_1$. Moreover, the similar triangles also give us that $\angle C_0C_1C_2 \cong \angle C_{i-1}C_iC_{i+1} = \pi - m\angle C_0AC_1 = \pi - \frac{\pi}{n}$ for any $i \in \{1, \dots, n-1\}$. This is sufficient to demonstrate that $C_0C_1 \dots C_n$ is similar to half of the k^2-*Perfect* $2n$-gon, with a ratio of $C_0C_1 : 1$.

We can compute C_0C_1 also by the law of cosines, getting

$$C_0C_1 = \sqrt{1^2 + k^4 - 2 \cdot 1 \cdot k^2 \cos\left(\frac{\pi}{n}\right)} = \sqrt{1 + k^4 - 2k^2 \cos\left(\frac{\pi}{n}\right)}.$$

Hence, $C_0C_1 \dots C_n$ has area

$$\frac{1 + k^4 - 2k^2 \cos\left(\frac{\pi}{n}\right)}{2} \cdot a(k^2, 2n).$$

Our construction of $C_0C_1 \dots C_n$ can be thought of as assembling the polygon from the n triangles $C_0AC_1, C_1AC_2, \dots, C_{n-1}AC_n$. These triangles are related to the ones left over from our $4n$-gon. For every triangle with edges k^{2i} and k^{2i+1} meeting at an angle $\pi - \frac{\pi}{2n}$, there is a triangle with edges k^{2i} and k^{2i+2} meeting at an angle $\frac{\pi}{n}$. Since any triangle ABC has area $\frac{1}{2}ab\sin C$, the ratio of the sum of areas of the triangles from the $4n$-gon to the sum of the areas of the triangles we just created is

$$\frac{2\sin\left(\pi - \frac{\pi}{2n}\right)}{k\sin\left(\frac{\pi}{n}\right)} = \frac{2\sin\left(\frac{\pi}{2n}\right)}{k\sin\left(\frac{\pi}{n}\right)}$$

(recall that we had $2n$ triangles in the first set but n triangles in the second set, hence the factor of 2). Therefore, the total area in the triangles left over from the $4n$-gon is

$$\frac{2\sin\left(\frac{\pi}{2n}\right)}{k\sin\left(\frac{\pi}{n}\right)} \cdot \frac{1 + k^4 - 2k^2\cos\left(\frac{\pi}{n}\right)}{2} \cdot a(k^2, 2n) = \frac{\sin\left(\frac{\pi}{2n}\right)\left(1 + k^4 - 2k^2\cos\left(\frac{\pi}{n}\right)\right)}{k\sin\left(\frac{\pi}{n}\right)} \cdot a(k^2, 2n).$$

Adding up, we get that

$$\frac{a(k, 4n)}{a(k^2, 2n)} = 1 + k^2 + 2k\cos\left(\frac{\pi}{2n}\right) + \frac{\sin\left(\frac{\pi}{2n}\right)\left(1 + k^4 - 2k^2\cos\left(\frac{\pi}{n}\right)\right)}{k\sin\left(\frac{\pi}{n}\right)}$$

Finally, we can plug in $k = 2$ and $n = 6$. This gives us

$$\begin{aligned} &5+4\cos\left(\frac{\pi}{12}\right) + \frac{\sin\left(\frac{\pi}{12}\right)\left(17 - 8\cos\left(\frac{\pi}{6}\right)\right)}{2\sin\left(\frac{\pi}{6}\right)} \\ &= 5 + 4 \cdot \frac{\sqrt{6}+\sqrt{2}}{4} + \frac{\sqrt{6}-\sqrt{2}}{4} \cdot \left(17 - 4\sqrt{3}\right) \\ &= \boxed{5 - \frac{25}{4}\sqrt{2} + \frac{25}{4}\sqrt{6}}. \end{aligned}$$

Note: One might expect at first glance that for fixed k and as n goes to infinity, this ratio would approach $(k+1)^2$. In the limit, $2n$-gon formed from taking every other vertex of the $4n$-gon will have side lengths that are $k+1$-times that of the k^2-*Perfect* $2n$-gon, and the leftover triangles look like their area will tend towards zero. However, we can see that their total area actually tends to $\frac{(k^2-1)^2}{2k}$, which grows faster than $(k+1)^2$ as k goes to infinity. As k goes to 1 i.e. as the polygons become regular, this quantity does approach 0.

Solution 2: The first solution gave us a decomposition of the k-*Perfect* $2n$-gon into $2n$ similar triangles (renaming k^2 to k). We can use this decomposition to write out an explicit formula for the area of the k-*Perfect* $2n$-gon.

Recall that we used n similar triangles to construct a polygon of area

$$\frac{1+k^2-2k\cos\left(\frac{\pi}{n}\right)}{2}a(k,2n).$$

Triangle C_0AC_1 had $AC_0 = 1$, $AC_1 = k$, and $m\angle C_0AC_1 = \frac{\pi}{n}$, so its area was

$$\frac{1}{2}\cdot 1\cdot k\cdot \sin\left(\frac{\pi}{n}\right) = \frac{k}{2}\sin\left(\frac{\pi}{n}\right).$$

The other triangles were similar, getting bigger in length by a factor of k each time, so the sum of the areas of the n triangles is

$$\begin{aligned}\frac{k}{2}\sin\left(\frac{\pi}{n}\right)(1+k^2+\cdots+k^{2(n-1)}) &= \frac{k}{2}\sin\left(\frac{\pi}{n}\right)\frac{k^{2n}-1}{k^2-1}\\ &= \frac{1+k^2-2k\cos\left(\frac{\pi}{n}\right)}{2}a(k,2n)\\ \implies a(k,2n) &= \frac{k\sin\left(\frac{\pi}{n}\right)(k^{2n}-1)}{(k^2-1)\left(1+k^2-2k\cos\left(\frac{\pi}{n}\right)\right)}.\end{aligned}$$

Hence, we have

$$\begin{aligned}\frac{a(k,4n)}{a(k^2,2n)} &= \frac{k\sin\left(\frac{\pi}{2n}\right)(k^{4n}-1)(k^4-1)\left(1+k^4-2k^2\cos\left(\frac{\pi}{n}\right)\right)}{k^2\sin\left(\frac{\pi}{n}\right)(k^{4n}-1)(k^2-1)\left(1+k^2-2k\cos\left(\frac{\pi}{2n}\right)\right)}.\\ &= \frac{\sin\left(\frac{\pi}{2n}\right)(k^4-1)\left(1+k^4-2k^2\cos\left(\frac{\pi}{n}\right)\right)}{k\sin\left(\frac{\pi}{n}\right)(k^2-1)\left(1+k^2-2k\cos\left(\frac{\pi}{2n}\right)\right)}.\end{aligned}$$

Plugging in $k = 2$, $n = 6$ yields the same answer as before.

Note: This formula recapitulates our earlier finding that this ratio grows as $O(k^3)$ as k grows large.

Aside: It may not be obvious that k-*Perfect* $2n$-gons exist for any integer n and positive real k. Here we give a constructive proof of their existence. In fact, we prove something stronger: given any positive real numbers $a_1, \ldots, a_n$, we construct an equiangular $2n$-gon $P_1P_2\ldots P_{2n}$ such that $P_iP_{i+1} = P_{n+i}P_{n+i+1} = a_i$ for all $i \in \{1, \ldots, n\}$.

Start with $P_1P_2\ldots P_{2n}$, a regular $2n$-gon with side length a_1. Now, translate the points P_3 through P_{n+2} by $a_2 - a_1$ units away from the other half of the points, in the direction parallel to P_2P_3 (if $a_2 - a_1 < 0$, move them towards the other points). This maintains all angles and all edge lengths, except that $P_2P_3 = P_{n+2}P_{n+3} = a_2$ now. Now do the same operation on the points $P_4, P_5, \ldots, P_{n+3}$, and so on. In the end, you will have constructed the desired polygon.