1. How many nonnegative integers less than 1000 have the property that the sum of their digits is a multiple of 3?

   Answer: 334

   Solution: These numbers correspond to the integers between 0 and 999 inclusive which are divisible by 3. This gives 334 because we include 0.

2. Compute the number of ways 6 girls and 5 boys can line up if all 11 people are distinguishable and no two girls stand next to each other.

   Answer: 86400

   Solution: Note that the lineup must be GBGBGBGBGBG. There are $6! \cdot 5! = \boxed{86400}$ ways that they can line up.

3. A certain high school has exactly 1000 lockers, numbered from 1 to 1000, all initially closed. Mark first opens every locker whose number has exactly 3 factors, starting with locker 4. Matt then opens every locker whose number is a power of 2, starting with locker 1. If Matt encounters a locker that Mark has already opened, he leaves it open. Compute the number of lockers that will be open when both Mark and Matt finish.

   Answer: 20

   Solution: Numbers with exactly three factors must be squares of primes (so the factors are 1, p, and p^2). Between 1 and 1000 there are 11 such numbers: 2^2, 3^2, 5^2, 7^2, 11^2, 13^2, 17^2, 19^2, 23^2, 29^2, 31^2. Furthermore, there are 10 powers of 2 between 1 and 1000: $2^0, 2^1, \ldots 2^9$. The number 4 is in each list, so there are a total of $\boxed{20}$ distinct lockers that Mark and Matt will open.

4. Andy has two identical cups, the first one is full of water and the second one is empty. He pours half the water from the first cup into the second, then a third of the water in the second into the first, then a fourth of the water from the first into the second and so on. Compute the fraction of the water in the first cup right before the 2015th transfer.

   Answer: $\mathbf{\frac{1008}{2015}}$

   Solution: Let the fraction of the water in the first cup after the ith step be a_i and the fraction of the water in the second cup at the ith step be b_i. We first show inductively that after an odd number of water transfers, both cups are half filled. Initially, we have $a_0 = 1$ and $b_0 = 0$. The first transfer transfers half the water from the first cup into the second so $a_1 = b_1 = \frac{1}{2}$. Now, suppose inductively that $a_{2k-1} = b_{2k-1} = \frac{1}{2}$. Then the $2k$th transfer will transfer $\frac{1}{2k+1}$ of the water in the second cup into the first cup. Thus, $a_{2k} = \left(1 + \frac{1}{2k+1}\right) \cdot \frac{1}{2}$ and $b_{2k} = \left(1 - \frac{1}{2k+1}\right) \cdot \frac{1}{2}$. Next, the $2k+1$th transfer will transfer $\frac{1}{2k+2}$ of the water in the first cup into the second cup. Thus,

$$\begin{aligned} a_{2k+1} &= \left(1 - \frac{1}{2k+2}\right) \cdot \left(1 + \frac{1}{2k+1}\right) \cdot \frac{1}{2} \\ &= \frac{2k+1}{2k+2} \cdot \frac{2k+2}{2k+1} \cdot \frac{1}{2} \\ &= \frac{1}{2} \end{aligned}$$

   Since $a_i + b_i = 1$ for all i, it follows that $b_{2k+1} = \frac{1}{2}$ also. Thus, by induction, after any odd number of water transfers, both cups are half filled.

So after the 2013th transfer, both cups are half filled. Next, the 2014th transfer will pour back $\frac{1}{2} \cdot \frac{1}{2015}$ of the water from the second cup into the first, so there is $\boxed{\frac{1008}{2015}}$ of the water in the first cup just before the 2015th transfer.

5. Alexander and Eliza are betting on a roll of a standard pair of 6-sided dice. Alexander bets that a sum of 12 will occur first and Eliza bets that two consecutive sums of 7 will appear first. They continue to roll the dice until one player wins. What is the probability that Alexander wins?

   Answer: $\mathbf{\frac{7}{13}}$

   Solution: Suppose that A is the event that Alexander wins. Let f be the sum of the dice on the first throw and s be the sum on the second throw. Thus,

$$P(A) = P(A|f = 12)P(f = 12) + P(A|f = 7)P(f = 7) + P(A|f \neq 12, 7)P(f \neq 12, 7)$$
$$P(A) = 1 \cdot \frac{1}{36} + P(A|f = 7)\frac{6}{36} + P(A)\frac{29}{36}$$

   Condition again,

$$\begin{aligned} P(A|f = 7) = &P(A|f = 7, s = 12)P(s = 12|f = 7) \\ &+ P(A|f = 7, s = 7)P(s = 7|f = 7) \\ &+ P(A|f = 7, s \neq 7, 12)P(s \neq 7, 12|f = 7) \end{aligned}$$

   Then,

$$P(A) = \frac{1}{36} + \left(1 \cdot \frac{1}{36} + 0 \cdot \frac{1}{6} + P(A)\frac{29}{36}\right)\frac{6}{36} + P(A)\frac{29}{36}$$

   Therefore, $P(A) = \boxed{\frac{7}{13}}$.

6. There are four seats arranged in a circle and a person is sitting on one of the seats. He rolls a standard six-sided die 6 times. For each roll of the die, if it lands on 4, he moves one seat clockwise. Otherwise, he moves k seats counterclockwise where k is the number he rolled. Compute the probability that he ends up on the same seat he originally started on.

   Answer: $\mathbf{\frac{61}{243}}$

   Solution: First, note that for each roll of the die, he has an equal probability of moving to any of the remaining seats. Let p_n be the probability that he returns to his seat after n rolls. Then $p_{n+1} = \frac{1}{3}(1 - p_n)$ and $p_1 = 0$. Therefore, we compute:

$$\begin{aligned} p_1 &= 0 \\ p_2 &= \frac{1}{3} \\ p_3 &= \frac{2}{9} \\ p_4 &= \frac{7}{27} \\ p_5 &= \frac{20}{81} \\ p_6 &= \boxed{\frac{61}{243}} \end{aligned}$$

7. For a positive integer n, let $f(n)$ denote the number of ones in the base 2 representation of n. For example, $f(13) = 3$ because $13 = 1101_2$. Compute the number of positive integers n that satisfy $n \le 2015$ and $f(n) \equiv f(n+1) \pmod 4$.

   Answer: 538

   Solution: We note that $f(n+1) - f(n)$ is determined by the number of ones at the end of the base 2 representation of n. If n ends in 0, then clearly $f(n+1) = f(n) + 1$. Otherwise, if n ends in 0 followed by k 1's (or if n in base 2 is just a string of k 1's), then $n+1$ must end in 1 followed by k 0's, so $f(n+1) = f(n) - k + 1$.

   We care about cases where $f(n+1) - f(n) \equiv 0 \pmod 4$. First, $f(n+1) - f(n) = 0$ if and only if n ends in $01 \Leftrightarrow n \equiv 1 \pmod 4$, so $n = 1, 5, \ldots, 2009, 2013$. There are $\frac{2013-1}{4} + 1 = 504$ such numbers. Similarly, $f(n+1) - f(n) = -4$ iff n ends in $011111 \Leftrightarrow n \equiv 31 \pmod{64}$, so $n = 31, 95, \ldots, 31 + 31 \cdot 64 = 2015$ for a total of 32 numbers. Finally, $f(n+1) - f(n) = -8$ iff n ends in 0 followed by 9 1's. This happens when n is $2^9 - 1 = 511$ or $2^{10} + 2^9 - 1 = 1535$. Since our numbers $< 2048 = 2^{11}$, they have at most 11 1's in their binary representations, so these are the only cases we need consider. So, report $504 + 32 + 2 = \boxed{538}$.

8. Find all ordered triples of positive integers (x, y, z) such that $x^2 + xz = y^2$ and $x + y + z = 40$.

   Answer: $\mathbf{(9, 15, 16), (2, 8, 30)}$

   Solution: Dividing through by z^2 we see that solving $x^2 + xz = y^2$ in the integers is equivalent to solving $x^2 + x = y^2$ in the rationals. Clearly, $(0, 0)$ is a solution. Observe that the line $y = mx$ intersects the locus of points given by $x^2 + x = y^2$ at $(0, 0)$ and exactly one other point. Also, by varying $m \in (-\infty, \infty)$ we obtain all intersection points; in other words, substituting $y = mx$ will give us all solutions. Specifically,

$$x^2 + x = m^2x^2 \Rightarrow x(x(1 - m^2) + 1) = 0 \Rightarrow x = \frac{1}{m^2 - 1}, y = \frac{m}{m^2 - 1} \text{ or } x = 0, y = 0.$$

   These solutions fix m to be rational in order for x, y to be rational. Then $m = \frac{p}{q}$ for p, q integers giving $x = \frac{q^2}{p^2 - q^2}, y = \frac{pq}{p^2 - q^2}$. Returning to the integers and the original equation of $x^2 + xz = y^2$ we clearly get solutions

$$x = kq^2, \quad y = kpq, \quad z = k(p^2 - q^2).$$

   Then $x + y + z = kp(p + q) = 40$. Also, we have $p > q > 0$ because $x, y, z > 0$.If $k = 1$, then $p > 4$ or else the sum is ≤ 32 and $p < 6$ or else the sum is ≥ 42. For $p = 5$ we get $q = 3$ yielding $(x, y, z) = (9, 15, 16)$. For $k = 2$ we get $p(p + q) = 20$. As before, $0 < q < p$ restricts p. This time, $p > 3$ and $p < 5$. For $p = 4$ we clearly get $q = 1$, which yields (along with $k = 2$) $(x, y, z) = (2, 8, 30)$. For $k \ge 4$ we get $p(p + q) \le 10$ which restricts $p < 3$ and $p > 2$. We conclude that there are exactly two solutions, and they are $\boxed{(9, 15, 16); (2, 8, 30)}$.

9. A sequence is formed of n 1s and m 0s in random order. A run is defined to be a consecutive string of 1s or 0s. What is the average number of runs?

   Answer: $\mathbf{1 + \frac{2nm}{n+m}}$

   Solution: Define

$$X_i = \begin{cases} 1 & \text{run of 1s starts at position } i, \\ 0 & \text{otherwise} \end{cases}$$

Let $R(1)$ be the number of runs of 1 and $R(0)$ the number of runs of 0. If there are n 1s and m 0s, then we can write $R(1) = \sum_{i=1}^{n+m} X_i$. So $E[R(1)] = \sum_{i=1}^{n+m} E[X_i]$. Next, $E[X_1] = \frac{n}{n+m}$ and for $1 < i \leq n+m$, $E[X_i] = P(0 \text{ in position i-1 and 1 in position i}) = \frac{m}{n+m}\frac{n}{n+m-1}$. Thus, the total expected number of runs is

$$E[R(1)]+E[R(0)] = \frac{n}{n+m}+(n+m-1)\frac{nm}{(n+m-1)(n+m)}+\frac{m}{n+m}+\frac{nm}{n+m} = \boxed{1+\frac{2nm}{n+m}}.$$

10. One 1×1 square tile and 115 1×5 tiles cover an entire 24×24 grid. How many positions can the square tile occupy?

Answer: 16

Solution: We color the grid with "colors" $0, 1, 2, 3, 4$. To define the coloring, we coordinatize the grid so that one side is the y-axis, a side that joins it is the x-axis, and that corner is the origin. Let the point (n, m) indicate the square in the grid given by coordinates $(n, m), (n, m+1), (n+1, m), (n+1, m+1)$. Consider the coloring $f(n, m) \equiv n - m \mod 5$. Clearly, any 5 consecutive squares in the vertical or horizontal directions are colored in the 5 distinct colors. Thus, any 1×5 tile covers each color exactly once. This means that, if a coloring exists, all colors appear 115 times except for one which appears 116 times; and the single square must color this last color. The function colors the main diagonal (which goes through the origin) of the grid with 0, and the colors $2, 3$ and $1, 4$ flip across the diagonal; then the number of squares colored $1, 2, 3, 4$ must be all equal and 0 is the odd one out. Thus, the single square must lie on a coordinates (m, n) such that $m - n \equiv 0 \mod 5$.

Next, we rotate the coloring; we could write it as the function given by $g(m, n) \equiv m + n \mod 5$. In this case, the other main diagonal is a solid color, and that color is the one with most squares. We know that $(0, 23)$ is on this diagonal, so the relevant color is 3. This means that the single square must lie on some coordinate (m, n) such that $m + n \equiv 3 \mod 5$. Combined with the previous restriction on (m, n) we get $2m \equiv 3 \mod 5 \Rightarrow m \equiv 4 \mod 5$ and this implies that $n \equiv 4 \mod 5$. Thus, there are currently $4 \cdot 4 = 16$ candidates for the position of the single square, because $m, n \equiv 4 \mod 5$ and $0 \leq m, n \leq 23$. We check that all these positions allow for a covering of the grid. By symmetry, there are only three cases to check; $(m, n) = (4, 4); (9, 4); (9, 9)$. In each case, we use 16 1×5 tiles to form a 9×9 square with the single square at its center. In each position case, punching out the 9×9 square leaves the 24×24 grid in a state that can be separated into several large rectangular subgrids each of which has one dimension divisible by 5. Specifically, $24 - 9 = 15$ so the $(4, 4)$ case leaves us with 9×15, 15×9 and 15×15 subgrids to cover. The $(9, 4)$ case leaves us with 5×24, 9×15 and 10×24 subgrids. The $(9, 9)$ case leaves us with 5×24, 5×10, 5×5, and 10×24 subgrids. Thus, all 16 potential positions allow for a full covering of the 24×24 grid by symmetry, giving us the answer of $\boxed{16}$.