1. Let $f(x) = x^4$ and let $g(x) = x^{-4}$. Compute $f''(2)g''(2)$.

   Answer: 15

   Solution: We note that $f''(x) = 12x^2$ and $g''(x) = 20x^{-6}$. Then $f''(x)g''(x) = 20 \cdot 12 \cdot x^{-4}$. Plugging in $x = 2$ we get $f''(2)g''(2) = \frac{12 \cdot 20}{16} = 3 \cdot 5 = \boxed{15}$.

2. There is a unique positive real number a such that the tangent line to $y = x^2 + 1$ at $x = a$ goes through the origin. Compute a.

   Answer: 1

   Solution: The slope of the tangent line is $2a$. The equation for the tangent line is $(y-(a^2+1)) = 2a(x - a)$. Setting $x = y = 0$ gives us $-a^2 - 1 = -2a^2$, which has solution $\boxed{a = 1}$.

3. Moor has \$1000, and he is playing a gambling game. He gets to pick a number k between 0 and 1 (inclusive). A fair coin is then flipped. If the coin comes up heads, Moor is given $5000k$ additional dollars. Otherwise, Moor loses $1000k$ dollars. Moor's happiness is equal to the log of the amount of money that he has after this gambling game. Find the value of k that Moor should select to maximize his expected happiness.

   Answer: $\mathbf{\frac{2}{5}}$

   Solution: Suppose that Moor chooses a value of k. We write down the expected value of Moor's happiness.

   If the coin comes up heads, Moor now has $1000 + 1000(5k) = 1000(5k + 1)$ dollars. If the coin comes up tails, Moor now has $1000 - 1000k = 1000(1 - k)$ dollars. Therefore, the expected value of Moor's happiness is

   $$H(k) = \frac{1}{2}\log(1000(5k + 1)) + \frac{1}{2}\log(1000(1 - k)).$$

   We want to maximize this. To do this, we differentiate, set the derivative equal to zero, and look for critical values. Here,

   $$H'(k) = \frac{1}{2}\left(\frac{5000}{1000(5k + 1)} - \frac{1000}{1000(1 - k)}\right) = \frac{1}{2}\left(\frac{5}{5k + 1} - \frac{1}{1 - k}\right) = 0$$

   when $5k + 1 = 5(1 - k)$, so $10k = 4$, and hence $k = \frac{2}{5}$ is the only critical value.

   The maximal value of $H(k)$ for $k \in [0, 1]$ must occur either at a critical value or an endpoint. Observe that among the three values $H(0)$, $H(1)$, and $H(\frac{2}{5})$, the largest is $H(\frac{2}{5})$. Therefore, Moor maximizes his happiness by selecting $k = \boxed{\frac{2}{5}}$.

4. The set of points (x, y) in the plane satisfying $x^{2/5} + |y| = 1$ form a curve enclosing a region. Compute the area of this region.

   Answer: $\mathbf{\frac{8}{7}}$

   Solution: The set of points satisfying the equation form a closed curve that encloses a region. Observe that this curve is preserved if we transform $x \mapsto -x$ or $y \mapsto -y$, so it is symmetric in all 4 quadrants. In particular, we can find the area in the first quadrant, where $x, y > 0$. In the quadrant, we can rewrite our equation as $y = 1 - x^{2/5}$. This curve intersects the coordinate axes at $(0, 1)$ and $(1, 0)$, and it is continuous, so the area is

   $$A = \int_0^1 1 - x^{2/5}\,dx = \frac{2}{7}.$$

The total area is therefore $4A = \boxed{8/7}$.

5. Compute the improper integral

$$\int_0^2 \left(\sqrt{\frac{4-x}{x}} - \sqrt{\frac{x}{4-x}}\right) dx.$$

Answer: 4

Solution 1: First of all, we note the many symmetries of the given expression. Specifically, we have $\sqrt{\frac{4-x}{x}}$ and we subtract its reciprocal. We also recall that square roots, when we take their derivative, give us their reciprocal. This inspires the guess that the function $f(x) = \sqrt{x}\sqrt{4-x}$ is somehow important to our integral. Indeed, we find that $f'(x) = \frac{1}{2}\left(\sqrt{\frac{4-x}{x}} - \sqrt{\frac{x}{4-x}}\right)$ so that $\int_0^2 \sqrt{\frac{4-x}{x}} - \sqrt{\frac{x}{4-x}}\, dx = 2\sqrt{x}\sqrt{4-x}\Big|_0^2 = 2(2-0) = \boxed{4}$.

Solution 2: Although solution 1 is, perhaps, the prettiest way of solving this problem, it is not necessarily easy to notice. A more direct approach uses a trig substitution. Specifically, noting the importance of $\frac{4-x}{x}$ and remembering the Pythagorean identity $\sin^2 x + \cos^2 x = 1$, it makes sense to try the substitution $x = 4\sin^2\theta$. Then $\sqrt{\frac{4-x}{x}} = \sqrt{\frac{\cos^2 x}{\sin^2 x}} = \cot x$. Also, $dx = 8\sin\theta\cos\theta\, d\theta$, $4\sin^2\theta = 0$ when $\theta = 0$ and $4\sin^2\theta = 2$ when $\theta = \frac{\pi}{4}$. The integral becomes

$$\int_0^{\frac{\pi}{4}} (\cot\theta - \tan\theta)\cdot 8\sin\theta\cos\theta\, d\theta = 8\int_0^{\frac{\pi}{4}} \cos^2\theta - \sin^2\theta\, d\theta = 8\int_0^{\frac{\pi}{4}} \cos 2\theta\, d\theta.$$

This last integral may be easily computed by the substitution $2\theta \mapsto \theta$:

$$8\int_0^{\frac{\pi}{4}} \cos 2\theta\, d\theta = 4\int_0^{\frac{\pi}{2}} \cos\theta\, d\theta = 4(\sin\theta)\Big|_0^{\frac{\pi}{2}} = 4(1-0) = \boxed{4}.$$

Solution 3: The simplest way to solve this problem is perhaps to write the integrand with a common denominator. This gives

$$\int_0^2 \frac{\sqrt{4-x}}{\sqrt{x}} - \frac{\sqrt{x}}{\sqrt{4-x}}\, dx = \int_0^2 \frac{(4-x)-x}{\sqrt{x}\sqrt{4-x}}\, dx = \int_0^2 \frac{4-2x}{\sqrt{4x-x^2}}\, dx.$$

Substitute $u = 4x - x^2$, $du = (4-2x)\, dx$. Then our integral becomes

$$\int_0^2 \frac{4-2x}{\sqrt{4x-x^2}}\, dx = \int_0^4 \frac{1}{\sqrt{u}}\, du = 2\sqrt{u}\Big|_0^4 = \boxed{4}.$$

6. Compute

$$\lim_{x\to\infty}\left[x - x^2 \ln\left(\frac{1+x}{x}\right)\right].$$

Answer: $\frac{1}{2}$

Solution: We rewrite this limit in a form that allows us to apply L'Hôpital's Rule. That is,

$$\lim_{x\to\infty}\left[x - x^2\ln\left(\frac{1+x}{x}\right)\right] = \lim_{x\to\infty}\frac{\frac{1}{x} - \ln\left(\frac{1+x}{x}\right)}{\frac{1}{x^2}}$$

$$= \lim_{x\to\infty}\frac{-\frac{1}{x^2} - \frac{x}{1+x}(-\frac{1}{x^2})}{\frac{-2}{x^3}} \quad \text{by L'Hôpital's Rule}$$

$$= \lim_{x\to\infty}\frac{1}{2}\left(x - \frac{x^2}{1+x}\right) = \lim_{x\to\infty}\frac{1}{2}\left(\frac{x}{1+x}\right) = \lim_{x\to\infty}\frac{1}{2}\left(1 - \frac{1}{1+x}\right) = \frac{1}{2}(1-0) = \boxed{\frac{1}{2}}.$$

7. For a given $x > 0$, let a_n be the sequence defined by $a_1 = x$ for $n = 1$ and $a_n = x^{a_{n-1}}$ for $n \geq 2$. Find the largest x for which the limit $\lim_{n\to\infty} a_n$ converges.

Answer: $\boldsymbol{e^{1/e}}$

Solution: In order for $\lim_{n\to\infty} a_n$ to have a limit L, it must be that $x^L = L$, so that $x = L^{1/L}$. Otherwise, we would be able to extend the recurrence and converge to a different limiting value. Thus, we seek the maximum of the function $f(L) = L^{1/L}$. To do this, we solve $\frac{df}{dL} = 0$. Since

$$\frac{df}{dL} = \frac{d}{dL}e^{\frac{\ln L}{L}} = L^{1/L}\left(\frac{1}{L^2} - \frac{\ln L}{L^2}\right),$$

we see that $L = e$. Thus, the maximum value for x is $f(e) = \boxed{e^{1/e}}$. To be sure that this is a maximum, we check as follows:

$$\left.\frac{d^2f}{dL^2}\right|_e = L^{\frac{1}{L}-4}\left(-3L + \ln^2(L) + 2(L-1)\ln(L) + 1\right)\Big|_e = -e^{\frac{1}{e}-3} < 0.$$

8. Evaluate

$$\int_{-2}^{2}\frac{1+x^2}{1+2^x}\,dx.$$

Answer: $\boldsymbol{\frac{14}{3}}$

Solution: We substitute the variable x by $-x$ and add the resulting integral to the original integral to get

$$2I = \int_{-2}^{2}\frac{1+x^2}{1+2^x}\,dx + \int_{2}^{-2} -\frac{1+x^2}{1+2^{-x}}\,dx = \int_{-2}^{2}\frac{1+x^2}{1+2^x} + \frac{1+x^2}{1+2^{-x}}\,dx$$

$$= \int_{-2}^{2}\frac{1+x^2}{1+2^x} + \frac{(1+x^2)2^x}{1+2^x}\,dx = \int_{-2}^{2}\frac{(1+x^2)\cdot(1+2^x)}{1+2^x}\,dx = \int_{-2}^{2} 1 + x^2\,dx = 4 + \frac{16}{3} = \frac{28}{3}.$$

So the given integral is $I = \boxed{\frac{14}{3}}$.

Note that more generally, for even functions f, $\int_{-a}^{a}\frac{f(x)}{1+b^x}\,dx = \frac{1}{2}\int_{-a}^{a} f(x)\,dx$.

9. Let f satisfy $x = f(x)e^{f(x)}$. Calculate $\int_0^e f(x)\,dx$.

Answer: $e - 1$

Solution 1: First, we compute the antiderivative. Make the substitution $u = f(x)$, so hence $du = f'(x)\,dx$. Note that $f'(x) = \frac{d}{dx}xe^{-f} = e^{-f} - f'(x)xe^{-f}$, so $f'(x) = \frac{1}{e^f + x} = \frac{f}{x(1+f)}$. Thus,

$$f(x)\,dx = u\left(\frac{du}{\frac{u}{x(1+u)}}\right) = x(1+u)\,du = ue^u(1+u)\,du$$

$$\int f(x)dx = \int ue^u(1+u)du = e^u\left(u^2 - u + 1\right) = e^{f(x)}\left(f(x)^2 - f(x) + 1\right)$$

To conclude, when $x = 0$, $f(x) = 0$ and when $x = e$, $f(x) = 1$. Thus, $\int_0^e f(x)\,dx = e^1(1 - 1 + 1) - e^0(0 - 0 + 1) = \boxed{e - 1}$.

Solution 2: Note that f is monotonically increasing and is the inverse of the function $g(y) = ye^y$. Since $f(e) = 1$, the area under $f(x)$ from 0 to e is the area of the rectangle with vertices $(0,0), (e,0), (0,1), (e,1)$ minus the area to the left of $f(x)$ from 0 to 1, and the latter is just the integral of $g(y)$ from 0 to 1. So we have

$$\int_0^e f(x)dx = e - \int_0^1 g(y)dy = e - \int_0^1 ye^y dy$$

$$= e - [ye^y]_0^1 + \int_0^1 e^y dy = \int_0^1 e^y dy = [e^y]_0^1 = e - 1.$$

10. Given that $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$, compute the sum

$$\sum_{n=1}^{\infty} \frac{1}{2^n n^2}.$$

Answer: $\frac{\pi^2}{12} - \frac{\ln^2 2}{2}$

Solution: First of all, for the sake of clarity, I omit details about certain calculations which are justifiable so there is a more clear focus on the actual computation. Specifically, I take derivatives and integrals of series without explaining, and I integrate functions with removable singularities, but the ordinary student would not pay attention to these technical issues anyway. I proceed now:

The first step to obtaining any insight on this problem is to replace $\frac{1}{2^n}$ with x^n. This allows us to take derivatives, getting rid of powers of n in the denominator. Thus, we write $f(x) = \sum_{n=1}^{\infty} \frac{x^n}{n^2}$ and what we want to find is $f(\frac{1}{2})$ given that $f(0) = 0$ and $f(1) = \frac{\pi^2}{6}$. As mentioned before, we first take $f'(x) = \sum_{n=1}^{\infty} \frac{x^{n-1}}{n}$ and then we take $(xf'(x))' = \sum_{n=1}^{\infty} x^{n-1} = \frac{1}{1-x}$. By reintegrating, $xf'(x) = -\ln(1-x) + C$ but by plugging in $x = 0$ it is easy to check that $C = 0$. Then $f'(x) = -\frac{\ln(1-x)}{x}$. Because $f(0) = 0$, $f(x) = \int_0^x -\frac{\ln(1-t)}{t}dt$. Thus, the answer we are looking for is equal to $\int_0^{\frac{1}{2}} -\frac{\ln(1-t)}{t}dt$. This completes the first part of the solution. The second part consists of computing this integral.

We denote $I = \int_0^{\frac{1}{2}} -\frac{\ln 1-t}{t}dt$. There are two things we know about this integral: that finding the antiderivative, if it even exists, would be extremely challenging, and also a related formula

$\int_0^1 -\frac{\ln(1-t)}{t}dt = \frac{\pi^2}{6}$ which is given. Noting that $\frac{1}{2}$ is the midpoint of the interval $[0,1]$ in which the integral formula is relevant, we note that there are several transformations which give integrals on the interval $[\frac{1}{2},1]$. Specifically, the substitution of $x \mapsto 1-x$ yields $I = \int_{\frac{1}{2}}^1 -\frac{\ln t}{1-t}dt$. In addition, $I = \int_0^1 -\frac{\ln(1-t)}{t}dt - \int_{\frac{1}{2}}^1 -\frac{\ln(1-t)}{t}dt = \frac{\pi^2}{6} - \int_{\frac{1}{2}}^1 -\frac{\ln(1-t)}{t}dt$. Thus, we may write $2I = \frac{\pi^2}{6} + \int_{\frac{1}{2}}^1 \frac{\ln 1-t}{t} - \frac{\ln t}{1-t}dt$. The apparent symmetry of the integrand immediately brings to mind the function $g(x) = \ln x \ln(1-x)$ as a potential antiderivative: indeed, when we apply the product rule, we easily get $g'(x) = \frac{\ln 1-x}{x} - \frac{\ln x}{1-x}$. Thus, $2I = \frac{\pi^2}{6} + \ln t \ln(1-t)\Big|_{\frac{1}{2}}^1$. Because plugging in $t=1$ is undefined, we resort to using limits and easily obtain 0. Thus, $2I = \frac{\pi^2}{6} - \ln^2\frac{1}{2} = \frac{\pi^2}{6} - \ln^2 2$ and $I = \boxed{\frac{\pi^2}{12} - \frac{\ln^2\frac{1}{2}}{2}} = \boxed{\frac{\pi^2}{12} - \frac{\ln^2 2}{2}}$ are both correct and equally valid answers.

Note, with only a little more work (and some formalizing), we can obtain the more general result that $\sum_{n=1}^{\infty}\frac{x^n}{n^2} + \sum_{n=1}^{\infty}\frac{(1-x)^n}{n^2} = \frac{\pi^2}{6} - \ln x \ln(1-x)$ when $x \in (0,1)$.