Introduction

This Power Round is an exploration of numerical semigroups, mathematical structures which appear very naturally out of answers to simple questions. For example, suppose McDonald's sells Chicken McNuggets in boxes containing a, b, or c McNuggets; can you say which exact quantities of McNuggets you can and cannot buy? The same problem is also often stated in terms of stamps or coins of certain values.

You can imagine that solutions to this problem must have numerous practical applications. What is more surprising is that it also has some interesting applications to more advanced, very abstract mathematics. We won't be able to discuss that here, but be aware, as you work through these elementary tricks and techniques for understanding numerical semigroups, that the same tricks and techniques are being used at the cutting edge of research!

Defining numerical semigroups

We will develop two different definitions of numerical semigroups, each of which has its intuitive advantages, and prove that they are in fact the same. We will use $\mathbb{N}_0$ to refer to the set of nonnegative integers $0, 1, 2, \ldots$.

Here is our first definition: let $a_1, \ldots, a_n$ be a set of positive integers ($n \geq 2$) such that $\gcd(a_1, \ldots, a_n) = 1$. The *numerical semigroup generated by* $a_1, \ldots, a_n$ is the set $\{c_1a_1 + \cdots + c_na_n \mid c_1, \ldots, c_n \in \mathbb{N}_0\}$, which we sometimes refer to as $\langle a_1, \ldots, a_n \rangle$. For example, $\langle 4, 6, 9 \rangle$ is the set $\{0, 4, 6, 8, 9, 10, 12, 13, \ldots\}$, which contains the listed numbers along with all integers after 12.

1. (a) **[6]** (i) Compute all elements of the numerical semigroup $\langle 5, 7, 11, 16 \rangle$.
   (ii) Can this numerical semigroup be generated by a set of fewer than 4 integers? Prove your answer.
   (iii) Compute all elements of the numerical semigroup $\langle 3, 7, 8 \rangle$.
   (iv) Can this numerical semigroup can be generated by a set of fewer than 3 integers? Prove your answer.
   (b) **[4]** Prove that $\langle a_1, \ldots, a_n \rangle$ is "closed under addition"—that is, if $x, y \in \langle a_1, \ldots, a_n \rangle$, then $x + y \in \langle a_1, \ldots, a_n \rangle$.
   (c) **[8]** Prove that $\langle a_1, \ldots, a_n \rangle$ contains all but a finite number of the nonnegative integers. (Hint: you may use without proof the fact that if $\gcd(a_1, \ldots, a_n) = 1$, then there exist *possibly negative* integers $d_1, \ldots, d_n$ such that $d_1a_1 + \cdots + d_na_n = 1$.)

Here is our second definition: a *numerical semigroup* is any set $S \subseteq \mathbb{N}_0$ which satisfies all of the following three properties: (i) S contains 0, (ii) S is "closed under addition"—that is, for any $x, y \in S$, we have $x + y \in S$, and (iii) S contains all but a finite number of the nonnegative integers. In Problem 1, you showed that $\langle a_1, \ldots, a_n \rangle$ is indeed a numerical semigroup by this definition.

2. (a) **[8]** Prove that any numerical semigroup S, by this definition, is "generated by" a finite set $\{a_1, \ldots, a_n\}$—that is, it can be written in the form $\langle a_1, \ldots, a_n \rangle = \{c_1a_1 + \cdots + c_na_n \mid c_1, \ldots, c_n \in \mathbb{N}_0\}$ where $a_1, \ldots, a_n$ are positive integers with $\gcd(a_1, \ldots, a_n) = 1$.
   (b) **[8]** We say that $\{a_1, \ldots, a_n\}$ is a *minimal generating set* of S if S is generated by $\{a_1, \ldots, a_n\}$ and S cannot be generated by any set of positive integers with fewer than n elements. Prove that every numerical semigroup S has a unique minimal generating set.

If a is part of the minimal generating set of S, we say that a is a *generator* of S. This will be important later.

The genus and Frobenius number of a numerical semigroup

Now that you have two equivalent definitions of numerical semigroups to work with, we can start analyzing them in more detail. The *genus* of a numerical semigroup S is the number of positive integers not contained in S. For example, $\langle 4, 6, 9\rangle = \{0, 4, 6, 8, 9, 10, 12, 13, \dots\}$ has genus 6, because it does not contain $1, 2, 3, 5, 7$, or 11. The *Frobenius number* of a numerical semigroup S is the largest integer that S does not contain. For example, $\langle 4, 6, 9\rangle$ has Frobenius number 11. Given a numerical semigroup S, let $g(S)$ be its genus and $F(S)$ its Frobenius number (Note that F may be negative. Specifically, if S contains all the non-negative integers, then $F(S) = -1$).

3. (a) [**4**] Compute the genus and Frobenius number of (i) $\langle 5, 7, 11, 16\rangle$ and (ii) $\langle 3, 7, 8\rangle$.
   (b) [**8**] Prove that for any numerical semigroup S, we have $F(S) \leq 2g(S) - 1$.

The famous Chicken McNugget Theorem states that if McDonald's sells Chicken McNuggets in boxes of a or b McNuggets where $\gcd(a, b) = 1$, then the largest number of McNuggets one cannot buy is $ab - a - b$.

4. (a) [**1**] Restate the Chicken McNugget Theorem in terms of the numerical semigroup $\langle a, b\rangle$.
   (b) [**8**] Prove the Chicken McNugget Theorem. (Possible hint: consider the grid

$$\begin{pmatrix} 1 & 2 & \cdots & a \\ a+1 & a+2 & \cdots & 2a \\ \vdots & \vdots & \ddots & \vdots \\ (b-1)a+1 & (b-1)a+2 & \cdots & ba \end{pmatrix}.$$

   Cross out the numbers of McNuggets that you can buy. What do you notice? Try this with actual numbers in place of a, b if you're not comfortable.)
   (c) [**10**] Find, with proof, the genus of $\langle a, b\rangle$.

The multiplicity, Apéry set, and embedding dimension of a numerical semigroup

The multiplicity of a numerical semigroup S is the smallest positive integer it contains. For example, $\langle 4, 6, 9\rangle = \{0, 4, 6, 8, 9, 10, 12, 13, \dots\}$ has multiplicity 4. We refer to the multiplicity of S by $m(S)$.

The Apéry set of a numerical semigroup S is the set $A(S) = \{n \mid n \in S, n - m(S) \notin S\}$. For example, $\langle 4, 6, 9\rangle$ has Apéry set $\{0, 6, 9, 15\}$. Notice that $A(S)$ always contains 0.

5. (a) [**4**] Compute the multiplicity and the Apéry set of (i) $\langle 5, 7, 11, 16\rangle$ and (ii) $\langle 3, 7, 8\rangle$.
   (b) [**4**] Prove that if numerical semigroup S has multiplicity m, then $A(S)$ can be uniquely written in the form $\{0, k_1 m + 1, k_2 m + 2, \dots, k_{m-1} m + m - 1\}$ where $k_1, \dots, k_{m-1}$ are positive integers and $k_i m + i$ is the smallest element of S which has a remainder of i when divided by m. For example, $A(\langle 4, 6, 9\rangle) = \{0, 2 \cdot 4 + 1, 1 \cdot 4 + 2, 3 \cdot 4 + 3\}$. In the future, we will often refer to $k_1, \dots, k_{m-1}$ as the Apéry coefficients of S.
   (c) [**4**] Prove that S is generated by $(A(S) - \{0\}) \cup \{m\}$. (Note that this does not mean $(A(S) - \{0\}) \cup \{m\}$ is a minimal generating set of S—in fact, that is not the case for our favorite example $\langle 4, 6, 9\rangle$.)
   (d) [**3**] Write, with proof, the genus of S in terms of its Apéry coefficients.
   (e) [**3**] Write, with proof, the Frobenius number of S in terms of its Apéry coefficients.

Note that because S is generated by $(A(S) - \{0\}) \cup \{m\}$, different numerical semigroups must have different Apéry sets. Hence we can associate each S with a unique sequence of Apéry coefficients $k_1, \ldots, k_{m-1}$. The natural next question becomes: when can an arbitrary sequence of positive integers $k_1, \ldots, k_{m-1}$ be the Apéry set of a valid numerical semigroup?

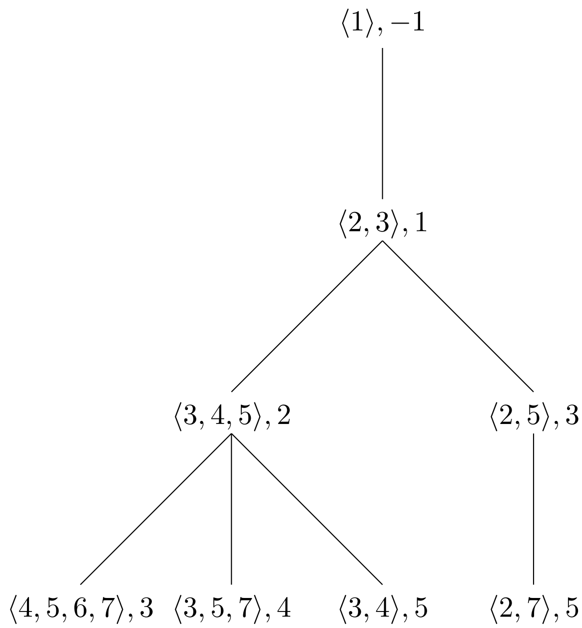
6. (a) **[4]** Suppose numerical semigroup S has Apéry coefficients $k_1, \ldots, k_{m-1}$. Prove that if $1 \leq i, j \leq m-1$ and $i + j < m$, then $k_i + k_j \geq k_{i+j}$. Also prove that if $1 \leq i, j \leq m-1$ and $i + j > m$, then $k_i + k_j + 1 \geq k_{i+j-m}$.
   (b) **[8]** Prove that if $k_1, \ldots, k_{m-1}$ satisfy the inequalities given in part a, there is a semigroup S with $k_1, \ldots, k_{m-1}$ as its Apéry coefficients.
   (c) **[8]** Find, with proof, in terms of g and m, the number of numerical semigroups S of genus g and multiplicity m satisfying $F(S) < 2m$.
   (d) **[8]** Prove that the number of numerical semigroups S of a fixed genus g (but any multiplicity) satisfying $F(S) < 2m(S)$ is a Fibonacci number.

The *embedding dimension* of a numerical semigroup S is the number of elements in its minimal generating set, which we call $e(S)$. Note that because S is generated by $(A(S) - \{0\}) \cup \{m\}$, we have $e(S) \leq m(S)$. If S is such that $e(S) = m(S)$, we say that S is a *maximal embedding dimension* numerical semigroup, or MED for short.

7. **[10]** Given a sequence of positive integers $k_1, \ldots, k_{m-1}$, give, with proof, necessary and sufficient conditions for $k_1, \ldots, k_{m-1}$ to be the Apéry coefficients of an MED numerical semigroup.

The semigroup tree

The semigroup tree is a systematic way of creating numerical semigroups. We start at level 0 of the tree, where we put the unique numerical semigroup of genus 0, that is, $\langle 1 \rangle = \mathbb{N}_0$. (By convention, $\mathbb{N}_0$ has Frobenius number -1.) If numerical semigroup S appears at level g, it has some number of *children* which appear at level $g + 1$. Each child is created by removing from the set S a single element n, with the condition that n is a generator (that is, an element of the minimal generating set of S) which is larger than the Frobenius number $F(S)$. Hence, we get the only child of $\langle 1 \rangle$ by removing 1, which results in $\langle 2, 3 \rangle$ of Frobenius number 1 at level 1. Now $2, 3$ are both larger than 1, so $\langle 2, 3 \rangle$ has two children at level 2: $\langle 3, 4, 5 \rangle$, which we get by removing 2, and $\langle 2, 5 \rangle$, which we get by removing 3. The first few levels of the tree are shown below. Each element is given in the format (minimal generating set, Frobenius number).



For convenience, when a semigroup has multiple children, we arrange them from left to right in increasing order of the size of the generator removed from the "parent".

8. (a) **[7]** Compute the next level of the tree, following the format given above. (So write each child in terms of its minimal generating set and give its Frobenius number.)
   (b) **[4]** Prove that as stated, the algorithm which generates the tree really does only create valid numerical semigroups, and that every numerical semigroup S appears exactly once in this tree (at level equal to its genus).
   (c) **[4]** Describe in general, with justification, all elements of the rightmost branch of the tree, including minimal generating set and Frobenius number.
   (d) **[4]** Describe in general, with justification, all elements of the leftmost branch of the tree, including minimal generating set and Frobenius number.

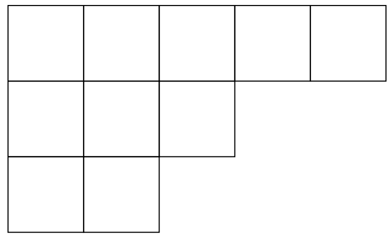
9. (a) **[6]** Suppose that S is not in the leftmost branch of the semigroup tree and that it has a child S'. Then answer—*proof not required*—the following in terms of the multiplicity, Frobenius number, and embedding dimension of S and S': (i) which generators of S are still generators of S'? (ii) Which generators of S are NOT generators of S'? (iii) Which generators of S' are NOT generators of S?
   (b) **[10]** Now, prove your answers to part a.

Weights

The *weight* of a numerical semigroup S is the sum of the positive integers not contained in S. For example, the weight of $\langle 4,6,9\rangle = \{0,4,6,8,9,10,12,13,\dots\}$ is $1+2+3+5+7+11 = 29$.

10. (a) **[2]** Compute the weight of (i) $\langle 5,7,11,16\rangle$ and (ii) $\langle 3,7,8\rangle$.
    (b) **[4]** Write, with proof, the weight of S in terms of its Apéry coefficients k_i. (You may leave your answer as a summation, but only over i.)
    (c) **[10]** Find, with proof, the weight of $\langle a,b\rangle$ in terms of a and b.

A *partition* of a positive integer n is a list of positive integers $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k$ such that $\lambda_1 + \lambda_2 + \cdots + \lambda_k = n$. For example, the distinct partitions of 4 are $4, 3+1, 2+2, 2+1+1$, and $1+1+1+1$. Each λ_i is called a *part* of a partition. Given a partition $\lambda = \lambda_1 + \cdots + \lambda_k$ of n, the *Ferrers-Young* diagram of λ consists of a row of λ_1 boxes, underneath which is a left-aligned row of λ_2 boxes, underneath which is a left-aligned row of λ_3 boxes, and so on. For example, the following figure is the Ferrers-Young diagram of the partition $5+3+2$ of 10.



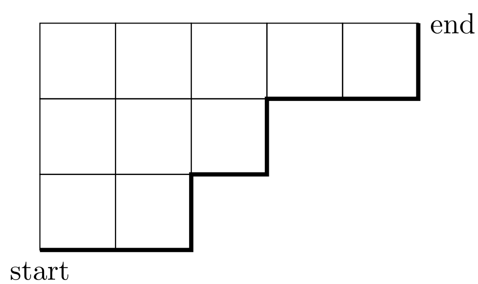
Given a box in a Ferrers-Young diagram, its associated *hook* is itself together with the boxes below it and the boxes to its right. The *size* or *length* of the hook is the number of boxes it contains. For example, the top left box in the Ferrers-Young diagram of $5+3+2$ is associated with a hook of length 7. All the hook lengths for the same partition are shown below.

7	6	4	2	1
4	3	1		
2	1			

The *hookset* of a partition λ, denoted H_λ, is the set of hook lengths which appear in the Ferrers-Young diagram of λ. For example, the hookset of $5+3+2$ is $\{1, 2, 3, 4, 6, 7\}$.

11. (a) **[8]** Let $p(x, y, z)$ be the number of partitions of x into at most y parts, each of size at most z. Prove that the number of numerical semigroups with genus g, multiplicity m, and weight w satisfying $m < F < 2m$ is exactly $p(w-(g-m+1), g-m+1, 2m-2-g)$.

    (b) **[10]** Prove that given any λ, the set $\mathbb{N}_0 \setminus H_\lambda$ is a numerical semigroup. (Possible hint: think of the Ferrers-Young diagram of λ as a partial grid whose edges one may walk along, and consider the walk starting at the bottom left corner and traversing the lower right edges of the diagram, as shown below.)



    (c) **[10]** Prove that given any numerical semigroup S, there exists a partition λ with $H_\lambda = \mathbb{N}_0 \setminus S$.