1. **Answer:** $\frac{250}{3}$

   Solution: This integral is equal to

$$\int_{-5}^{5} x + x^2 + x^3\,dx = \int_{-5}^{5} x^2\,dx = \left(\frac{5^3}{3} - \frac{(-5)^3}{3}\right) = \boxed{\frac{250}{3}}.$$

2. **Answer: 20**

   Solution: Clearly we want to maximize $f(x)$ when $\sin(x) \geq 0$ and minimize $f(x)$ when $\sin(x) < 0$. We do this by setting $f(x) = 5$ in the first case and $f(x) = -5$ in the second case. Noting that the bounds of integration cover precisely one full period of sin, we see that the integral becomes equivalent to twice the integral of $5\sin(x)$ over the half period where $\sin(x) \geq 0$. This results in $\boxed{20}$.

3. **Answer:** $\frac{5}{2}\ln\frac{8}{3}$

   Solution: Note that we can write the integral as

$$\int_{2^5}^{3^5} \frac{1}{x^{3/5}(x^{2/5}-1)}\,dx.$$

   We solve via u-substitution. Let $u = x^{2/5} - 1$:

$$du = \frac{2}{5}x^{-3/5}\,dx \implies dx = \frac{5}{2}x^{3/5}\,du.$$

   The integral becomes

$$\frac{5}{2}\int_{2^2-1}^{3^2-1} \frac{x^{3/5}}{x^{3/5}\cdot u}\,du = \frac{5}{2}\int_{3}^{8}\frac{1}{u}\,du,$$

   which evaluates to

$$\frac{5}{2}(\ln 8 - \ln 3) = \boxed{\frac{5}{2}\ln\frac{8}{3}}.$$

4. **Answer:** $\frac{2+\sqrt{3}}{2}$

   Solution: We want to minimize the distance between the points (a^2, a) and $(2, 1)$. We can equivalently minimize the square of the distance between those two points, which is

$$(2-a^2)^2 + (1-a)^2 = a^4 - 3a^2 - 2a + 5.$$

   The derivative of this function is $4a^3 - 6a - 2$, which can be factored as $2(a+1)(2a^2 - 2a - 1)$. The roots of this cubic are therefore $a = -1, \frac{1\pm\sqrt{3}}{2}$. Two of the roots are negative and therefore invalid, so therefore $a = \frac{1+\sqrt{3}}{2}$ and

$$a^2 = x = \boxed{\frac{2+\sqrt{3}}{2}}$$

5. **Answer: 1**

   Solution: We begin by observing that due to the inverse function rule, the first three derivatives of g are determined by the first three derivatives of f. Additionally, $f(0) = g(0) = 0$. Let $\hat{f}(x)$ and $\hat{g}(x) = \hat{f}^{-1}(x)$ be new functions whose first three derivatives at zero equal those of f and g respectively.

   By Taylor Series expansion, we see that $\hat{f}(x) = -\ln(1-x)$ is a suitable choice. Then $\hat{g}(x) = 1 - e^{-x}$ and $\hat{g}^{(3)}(0) = g^{(3)}(0) = e^0 = \boxed{1}$.

6. **Answer:** $e^{-1/3}$

   Solution: We can approximate $\sin x$ and $\cos x$ by their Taylor series.

$$\sin x \approx x - \frac{x^3}{6}, \quad \cos x \approx 1 - \frac{x^2}{2}$$

Substituting the limit becomes

$$\lim_{x\to 0}\left(1-\frac{x^2}{6}\right)^{2/x^2}=\lim_{x\to\infty}\left(1+\frac{1}{6x^2}\right)^{-2x^2}=\lim_{x\to\infty}\left(1+\frac{1}{x^2}\right)^{-x^2/3}=\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^{-x/3}=\boxed{e^{-1/3}}$$

because $e=\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x$.

7. **Answer: $\mathbf{12x^2-24x+26-26e^{-x}}$**

Solution: First differentiate the equation with respect to x:

$$g(x)+\int_0^x g(t)\,dt=4x^3+2x.$$

Differentiate again to obtain

$$g'(x)+g(x)=12x^2+2.$$

A particular solution $12x^2-24x+26$ can be found using the method of undetermined coefficient, so the general solution will be

$$g(x)=12x^2-24x+26+Ce^{-x}$$

for some constant C. By substituting $x=0$ into the first equation, we see that $g(0)=0$. We therefore find that $C=-26$, making the answer $\boxed{12x^2-24x+26-26e^{-x}}$.

8. **Answer: $\frac{\pi\ln 2}{4}$**

Solution: Substitute $x=2\tan\theta$ to get

$$\int_0^\infty\frac{\ln x}{x^2+4}\,dx=\frac{1}{2}\int_0^{\pi/2}\ln(2\tan\theta)\,d\theta=\frac{1}{2}\cdot\frac{\pi}{2}\ln 2+\frac{1}{2}\int_0^{\pi/2}\ln(\tan\theta)\,d\theta.$$

We will now show that this final integral is zero by substituting $\theta=\pi/2-\phi$ to yield

$$\begin{aligned}\int_0^{\pi/2}\ln(\tan\theta)\,d\theta&=-\int_{\pi/2}^{0}\ln\left(\tan\left(\frac{\pi}{2}-\phi\right)\right)\,d\phi\\&=\int_0^{\pi/2}\ln\left(\frac{1}{\tan\phi}\right)\,d\phi=-\int_0^{\pi/2}\ln(\tan\phi)\,d\phi,\end{aligned}$$

which gives us what we wanted, so the answer is $\boxed{\frac{\pi\ln 2}{4}}$.

9. **Answer: $(1/2, e^{-3/4})$**

Solution: Take the logarithm and approximate using Stirling's approximation[1].

Stirling's approximation says that $\ln(n!)\approx n\ln n-n$ in the limit of large n. Using this, we have

$$\begin{aligned}\ln\left(\frac{\sqrt[n^2]{1!2!\cdots n!}}{n^\alpha}\right)&=\frac{1}{n^2}\ln(1!2!\cdots n!)-\alpha\ln n=\frac{1}{n^2}\sum_{k=1}^{n}\ln(k!)-\alpha\ln n\\&\approx\frac{1}{n^2}\sum_{k=1}^{n}(k\ln k-k)-\alpha\ln n=\frac{1}{n^2}\sum_{k=1}^{n}(k\ln k)-\frac{1}{n^2}\frac{n(n+1)}{2}-\alpha\ln n\end{aligned}$$

Approximate $n(n+1)$ with n^2 and approximate the infinite sum by an integral

$$\approx\frac{1}{n^2}\int_1^n x\ln x\,dx-\frac{1}{2}-\alpha\ln n$$

[1] `http://en.wikipedia.org/wiki/Stirling's_approximation`

Integrating by parts

$$\approx \frac{1}{n^2}\left(\frac{n^2}{2}\ln n - \frac{n^2}{4} + \frac{1}{4}\right) - \frac{1}{2} - \alpha \ln n \approx \frac{1}{2}\ln n - \frac{3}{4} - \alpha \ln n.$$

Therefore,

$$\lim_{n\to\infty} \ln\left(\frac{\sqrt[n^2]{1!2!\cdots n!}}{n^\alpha}\right) = \lim_{n\to\infty}\left[\left(\frac{1}{2}-\alpha\right)\ln n - \frac{3}{4}\right],$$

which is finite only when $\frac{1}{2} - \alpha = 0$, in which case $\alpha = \frac{1}{2}$ and the limit evaluates to $\frac{3}{4}$. Therefore, we wish to compute

$$\lim_{n\to\infty} \frac{\sqrt[n^2]{1!2!\cdots n!}}{n^{1/2}} = \exp\left[\lim_{n\to\infty} \ln\left(\frac{\sqrt[n^2]{1!2!\cdots n!}}{n^\alpha}\right)\right] = \boxed{e^{3/4}}.$$

10. **Answer: 1/4**

Solution: Consider the following expression

$$\int_0^1 (f(x)-1)\left(f(x)+\frac{1}{2}\right)^2 dx.$$

Since $f(x) \leq 1$ this expression is less than or equal to 0. Meanwhile expanding the integrand gives

$$(f(x)-1)\left(f(x)+\frac{1}{2}\right)^2 = f(x)^3 - \frac{3}{4}f(x) - \frac{1}{4},$$

so its integral is

$$\begin{aligned}\int_0^1 (f(x)-1)\left(f(x)-\frac{1}{2}\right)^2 dx &= \int_0^1 f(x)^3\,dx - \frac{3}{4}\int_0^1 f(x)\,dx - \frac{1}{4}\int_0^1 dx \\ &= \int_0^1 f(x)^3\,dx - \frac{1}{4},\end{aligned}$$

proving that the answer is at most 1/4. Equality occurs when

$$f(x) = \begin{cases} -1/2 & \text{if } 0 \leq x \leq 2/3 \\ 1 & \text{if } 2/3 < x \leq 1 \end{cases},$$

so $\boxed{1/4}$ is indeed the maximum.