1. Answer: \(2 + 2\sqrt{2}\)

The path made generates a regular octagon with side length 1, since the exterior angle of the octagon is 45°. Notice that by inscribing the octagon in a square of side length \(1 + \sqrt{2}\), we can easily calculate that the octagon has area \((1 + \sqrt{2})^2 - 4 \left( \frac{1}{4} \right) = 2 + 2\sqrt{2}\).

2. Answer: \(\frac{33\sqrt{3}}{28}\)

Looking at cyclic quadrilaterals \(ABCD\) and \(ACDF\) tells us that \(m\angle ACD = m\angle ADC\), so \(\triangle ACD\) is equilateral and \(m\angle DEA = 120^\circ\). Now, if we let \(m\angle EAD = \theta\), we see that \(m\angle CAB = 60^\circ - \theta \Rightarrow m\angle ACB = \theta \Rightarrow \triangle AED \sim \triangle CBA\). Now all we have to do is calculate side lengths. After creating some \(30^\circ - 60^\circ - 90^\circ\) triangles, it becomes evident that \(AC = \sqrt{3}\). Now let \(AB = x\), so \(BC = 2x\).

By applying the Law of Cosines to triangle \(ABC\), we find that \(x^2 = \frac{3}{7}\). Hence, the desired area \((ABCDE) = (ACD) + 2(ABC) = \left(\sqrt{3}\right)^2 + 2 \cdot \frac{1}{2}x(2x)(\sin 120^\circ) = \frac{33\sqrt{3}}{28}\).

3. Answer: 15

After some angle chasing, we find that \(m\angle DBF = m\angle DFB = 75^\circ\), which implies that \(DF = DB\). Hence the desired perimeter is equal to \(AF - BF + AE + FE = 20 - BF + FE\).

By the law of sines, \(\frac{FE}{\sin 30^\circ} = \frac{10}{\sin 75^\circ} \Rightarrow FE = \frac{5}{\sqrt{6} - \sqrt{2}} = 5\sqrt{6} - 5\sqrt{2} \). Now, to find \(BF\), draw the altitude from \(O\) to \(AB\) intersecting \(AB\) at \(P\). This forms a \(30^\circ - 60^\circ - 90^\circ\) triangle, so we can see that \(AP = 5\sqrt{3}/2 = \frac{10 - BF}{2} \Rightarrow BF = 10 - 5\sqrt{3}\). Hence, the desired perimeter is \(20 + (5\sqrt{6} - 5\sqrt{2}) - (10 - 5\sqrt{3}) = 10 - 5\sqrt{2} + 5\sqrt{3} + 5\sqrt{6}\), so the answer is \(10 - 5 + 5 + 5 = 15\).

4. Answer: \(5 + \sqrt{19}\)

Rotate the figure around \(A\) by \(60^\circ\) so that \(C\) coincides \(B\). Let \(B', C', D', E'\) be the points corresponding to \(B, C, D, E\) in the rotated figure. Since \(\angle E'AD = \angle E'AC' + \angle C'AD = \angle EAC + \angle BAD = 30^\circ = \angle EAD\), \(E'A = EA\) and \(DA = D'A\), one has \(E'D = ED\). So \(BC = BD + DE + EC\) can be found if we know \(E'D\). But \(E'D = \sqrt{E'B^2 + BD^2 - 2 \cdot E'B \cdot BD \cdot \cos 120^\circ} = \sqrt{19}\), so \(BC = 2 + \sqrt{19} + 3 = 5 + \sqrt{19}\).

5. Answer: \(10\sqrt{2}\)
We have $\triangle ADE \sim \triangle CBE$, and their length ratio is $AD : CB = 1 : 2$. Let $AE = p$ and $DE = q$. Then we have $AB = BE - AE = 2DE - AE = 2q - p$ and $CD = 2p - q$. Solving for $p$ and $q$, we have $p = 4$ and $q = 5$. Similarly we have $FC = 8$ and $FD = 10$. Let $\angle B = \theta$. Then $\angle FDE = \pi - \theta$. Apply the Law of Cosines to $\triangle EBF$ to get

$$EF^2 = BE^2 + BF^2 - 2BE \cdot BF \cdot \cos \theta = 10^2 + 20^2 - 2 \cdot 10 \cdot 20 \cos \theta = 500 - 400 \cos \theta$$

and to $\triangle EDF$ to get

$$EF^2 = DE^2 + DF^2 + 2 \cdot DE \cdot DF \cos \theta = 5^2 + 10^2 - 2 \cdot 5 \cdot 10 \cos \theta = 125 + 100 \cos \theta.$$ Solving for $EF^2$, we get $EF^2 = 200$.

6. Answer: $\pi - \tan^{-1}\left(\frac{1}{4}\right)$ (or $\pi/2 + \tan^{-1} d$) or other equivalent form

Construct points $C_1, C_2, C_3, \ldots$ on $l_1$ progressing in the same direction as the $A_i$ such that $C_1 = A_1$ and $C_n C_{n+1} = 1$. Thus we have $C_1 = A_1$, $C_3 = A_2$, $C_5 = A_3$, etc., with $C_{2n-1} = A_n$ in general. We can write $\angle A_i B_i A_{i+1} = \angle C_{2i-1} B_i C_{2i+1} = \angle C_i B_i C_{2i+1} - \angle C_i B_i C_{2i-1}$. Observe that $\angle C_i B_i C_k$ (for any $k$) is a right triangle with legs of length $d$ and $k - i$, and $\angle C_i B_i C_k = \tan^{-1} \frac{k-i}{d}$. So $\angle C_i B_i C_{2i+1} - \angle C_i B_i C_{2i-1} = \tan^{-1} \frac{i+1}{d} - \tan^{-1} \frac{i-1}{d}$. The whole sum is therefore

$$\sum_{i=1}^{\infty} \left( \tan^{-1} \frac{i+1}{d} - \tan^{-1} \frac{i-1}{d} \right)$$

which has $n$th partial sum

$$\tan^{-1} \frac{n+1}{d} + \tan^{-1} \frac{n}{d} - \tan^{-1} \frac{1}{d}$$
so it converges to \( \pi - \tan^{-1} \frac{1}{2} \).

7. Answer: \( \sqrt{5} \)

Rotate triangle \( APB \) around \( A \) by 90 degrees as in the given figure. Let \( P' \) and \( B' \) be the rotated images of \( P \) and \( B \) respectively. Then we have \( B'P' = BP \), \( P'P = \sqrt{2}AP \) so
\[ \sqrt{2}AP + BP + CP = CP = PP' + P'B' \leq CB' = \sqrt{5}. \]

8. Answer: \( \frac{\pi}{6} \)

Consider the cube to be of side length 2 and divide the answer by 4 later. Set the coordinates of the vertices of the cube to be \((\pm 1, \pm 1, \pm 1)\). Then the plane going through an equilateral triangle can be described by the equation \( x + y + z = 1 \). The distance to the plane from the origin is \( \frac{1}{\sqrt{3}} \), as \( \left( \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}} \right) \) is the foot of the perpendicular from \((0, 0, 0)\). Thus the radius of the circle is \( \sqrt{1 - \left( \frac{1}{\sqrt{3}} \right)^2} = \frac{\sqrt{2}}{3} \), so the area is \( \frac{2}{3} \pi \). In the case of the unit cube we should divide this by 4 to get the answer \( \frac{\pi}{6} \).

9. Answer: \( \frac{49}{390} \)

First we shall find \( \frac{\text{area}(\triangle ADC)}{\text{area}(\triangle ABC)} \): Since \( \triangle A'RB \sim C'C'B \) and \( A'B = \frac{1}{3}BC \), it follows that \( A'R = \frac{1}{3}CS \). Then \( \text{area}(\triangle AA'C') = \frac{1}{2}A'R \times AC' = \frac{1}{2}(\frac{1}{3}CS)(\frac{1}{3}AB) = \frac{1}{18}(\frac{1}{3}CS \times AB) = \frac{1}{18} \text{area}(\triangle ABC) \). Similarly, \( \text{area}(\triangle AA'C') = \frac{1}{2}AP \times A'C = \frac{1}{2}AP(\frac{1}{3}BC) = \frac{1}{6} \text{area}(\triangle ABC) \). So \( \frac{\text{area}(\triangle AA'C')}{\text{area}(\triangle ABC)} = \frac{1}{12} \). Since \( \triangle AA'C' \) and \( \triangle AA'C \) share the same base, \( \frac{CT}{QC} = \frac{1}{12} \). Since \( \triangle C'TD \sim \triangle CQD \), \( \frac{C'T}{C'D} = \frac{1}{12} \). Using similar arguments, since \( AC' = \frac{1}{3}AB \), \( \text{area}(\triangle AC'C) = \frac{1}{3} \text{area}(\triangle ABC) \). Since \( CD = \frac{12}{13}C'C \), \( \text{area}(\triangle ADC) = \frac{12}{13} \times \frac{1}{3} \text{area}(\triangle ABC) \). Using the same technique, we can find \( \frac{\text{area}(\triangle ABC)}{\text{area}(\triangle ABC)} \) and \( \frac{\text{area}(\triangle BFC)}{\text{area}(\triangle ABC)} \). We will just briefly outline the remaining process: \( \frac{\text{area}(\triangle A'B'C')}{\text{area}(\triangle ABC)} = \frac{\frac{1}{2} \times \frac{1}{2}}{\frac{3}{4}} = \frac{1}{3} \). So \( \frac{A'B'}{BP} = \frac{1}{3} \). Then \( \text{area}(\triangle BB'C) = \frac{1}{2} \text{area}(\triangle ABC) \), so \( \text{area}(\triangle BFC) = \frac{4}{5} \text{area}(\triangle BB'C) = \frac{2}{3} \text{area}(\triangle ABC) \).
 Likewise, \( \frac{\text{area}(\triangle BB'A')}{\text{area}(\triangle BB'A)} = \frac{\frac{1}{2} \times \frac{1}{2}}{3} = \frac{1}{6} \). So \( \frac{\text{area}(\triangle AA'B)}{\text{area}(\triangle ABC)} = \frac{1}{6} \). Then \( \text{area}(\triangle AA'B) = \frac{1}{6} \text{area}(\triangle ABC) \), so \( \text{area}(\triangle AEB) = \frac{5}{6} \text{area}(\triangle AA'B) = \frac{5}{36} \text{area}(\triangle ABC) \).

Then \( \frac{\text{area}(\triangle DEF)}{\text{area}(\triangle ABC)} = 1 - \frac{\text{area}(\triangle ADC)}{\text{area}(\triangle ABC)} - \frac{\text{area}(\triangle BFC)}{\text{area}(\triangle ABC)} - \frac{\text{area}(\triangle AEB)}{\text{area}(\triangle ABC)} = 1 - \frac{12}{39} - \frac{2}{5} - \frac{1}{6} = \frac{49}{390} \).

10. Answer: \( 2\sqrt{43} \)

We claim that in general, the answer is \( \sqrt{\frac{2}{3} (a^2 + b^2 + c^2 + 4\sqrt{3}S)} \), where \( S \) is the area of \( ABC \).

Suppose that \( PQR \) is an equilateral triangle satisfying the conditions. Then \( \angle BPC = \angle CQA = \angle ARB = 60^\circ \). The locus of points satisfying \( \angle BXC = 60^\circ \) is part of a circle \( O_a \). Draw \( O_b \) and \( O_c \) similarly. These three circles meet at a single point \( X \) inside the triangle, which is the unique point satisfying \( \angle BXC = \angle CXA = \angle AXB = 120^\circ \). Then the choice of \( P \) on \( O_a \) determines \( Q \) and \( R \): those two points should also be on \( O_b \) and \( O_c \), respectively, and line segments \( PCQ \) and \( PBR \) should form sides of the triangle. Now one should find the maximum of \( PQ \) under these conditions. Note that \( \angle BPX \) and \( \angle BRX \) do not depend on the choice of \( P \), so triangle \( PXR \) has the same shape regardless of our choice. In particular, the ratio of \( PX \) to \( PR \) is constant, so \( PR \) is maximized when \( PX \) is the diameter of \( O_a \). This requires \( PQ, QR, RP \) to be perpendicular to \( XC, XA, XB \) respectively.

From this point there may be several ways to calculate the answer. One way is to observe that \( PQ = \frac{2}{\sqrt{3}}(AX + BX + CX) \) by considering \( (PQR) = (PXQ) + (QXR) + (RXP) \). \( AX + BX + CX \) can be computed by the usual rotation trick for the Fermat point: rotate \( \triangle BXA 60^\circ \) around \( B \) to \( \triangle BX'A' \). Observe that \( \triangle BXX' \) is equilateral, and so \( A', X', X, \text{ and } C \) are collinear. Hence, \( A'C = AX + BX + CX \), and we can apply the Law of Cosines to \( \triangle A'B'C \) to get that \( A'C^2 = a^2 + a^2 - 2ac \cos (B + 60^\circ) = a^2 + c^2 + 2ac \sin 60^\circ \sin B - 2ac \cos 60^\circ \cos B = a^2 + c^2 + 2S\sqrt{3} - 1/2(a^2 + c^2 - b^2) = \frac{a^2 + b^2 + c^2}{2} + 2S\sqrt{3} \implies PQ = \frac{2}{\sqrt{3}} (a^2 + b^2 + c^2 + 4\sqrt{3}S) \) (where \( S \) is again the area of \( ABC \)). Plugging in our values for \( a, b, \text{ and } c \), and using Heron’s formula to find \( S = \sqrt{10 \times 5 \times 3 \times 2} = 10\sqrt{3} \), we can calculate \( PQ = 2\sqrt{43} \).