1. **Answer: $312 - 180\sqrt{3}$**

   First let a be the length of AE. Then $CE = a/\sqrt{2}$, $BE = 1 - a/\sqrt{2}$ so $AE^2 = a^2 = 1 + BE^2 = 2 - \sqrt{2}a + a^2/2$. Solving it gives $a^2 + 2\sqrt{2}a - 4 = 0$, $(a+\sqrt{2})^2 = 6$ so $a = \sqrt{6} - \sqrt{2}$.
   Next let b be the length of IJ. Then AIJ is equilateral so $AJ = b$. Also $JE = 2/\sqrt{3}b$, so $AE = a = \frac{2+\sqrt{3}}{\sqrt{3}}b$, $b = (2-\sqrt{3})(\sqrt{3})(\sqrt{6}-\sqrt{2}) = \sqrt{2}(9-5\sqrt{3})$. Squaring it gives $312 - 180\sqrt{3}$.

2. **Answer: 1, −1**

   The whole equation is $\equiv 0 \pmod 3$, so x^3+6x^2+2x-6 should be 3 or -3. The equation $(x^3+6x^2+2x-6)^2 = 3^2$ can be rewritten using difference of squares as $(x-1)(x^2+7x-9)(x+1)(x^2+5x-3) = 0$, so only 1 and -1 work for x.

3. **Answer: 12**

   After dividing the equation by $4x^2$, we can re-write it as

   $$a\left(\frac{x}{2}+\frac{1}{2x}\right)^2 + \left(\frac{x}{2}+\frac{1}{2x}\right) - a = b.$$

   Set $y = \frac{x}{2} + \frac{1}{2x}$, which has range $(-\infty, -1] \cup [1, \infty)$. Therefore, we need all b in $(-2, 2)$ such that b is in the range of $f(y) = ay^2 + y - a$ for the domain $y \in (-\infty, -1] \cup [1, \infty)$. The vertex of this parabola lies at $y = -\frac{1}{2a} \in (-1/4, -1/12)$, so the desired range is just all values greater than $f(-1) = -1$. Hence, A is the set of all points where $-1 < b < 2$ and $2 < a < 6$, so the area is 12.

4. **Answer: 0**

   A polynomial $p(x)$ has a multiple root at $x = a$ if and only if $x - a$ divides both p and p'. Continuing inductively, the nth derivative $p^{(n)}$ has a multiple root b if and only if $x - b$ divides $p^{(n)}$ and $p^{(n+1)}$. Since $f(x)$ has 1 as a root with multiplicity 4, $x-1$ must divide each of f, f', f'', f'''. Hence $f'''(1) = 0$. Similarly, $x-2$ divides each of f, f', f'' so $f''(2) = 0$ and $x-3$ divides each of f, f', meaning $f'(3) = 0$. Hence the desired sum is 0.

5. **Answer: $P(x) = 1 - x^2$**

   First suppose $P(x)$ is constant or linear. Then we have $P(2010) + P(2012) = 2P(2011)$, which is a contradiction because the left side is congruent to 1 (mod 3) and the right is congruent to 0 (mod 3). So P must be at least quadratic. The space of quadratic polynomials in x is spanned by the polynomials $f(x) = 1$, $g(x) = x$, and $h(x) = x^2$. Applying each of these to 2010, 2011, and 2012, we have the mod 3 equivalences:

   $$f(2010, 2011, 2012) \equiv (1, 1, 1)$$
   $$g(2010, 2011, 2012) \equiv (0, 1, 2)$$
   $$h(2010, 2011, 2012) \equiv (0, 1, 1)$$

   Subtracting the third row from the first, we have $P(x) = f(x) - h(x) = 1 - x^2$, giving $P(2010, 2011, 2012) \equiv (1, 0, 0) \pmod 3$, as desired. Uniqueness follows from the observation that the three vectors above form a basis for $(\mathbb{Z}/3\mathbb{Z})^3$.

6. **Answer: 10**

   Consider the graphs of $y = t^3 - 12t^2 + 21t$ and $y = p (p \leq 0)$. These two graphs intersect at three points (counting multiplicity) if and only if there are three nonnegative x, y, z satisfying $xyz = p$. In order for these two to intersect at three points, p should lie between the local maximum and the local minimum of the cubic function $y = t^3 - 12t^2 + 21t$, so the maximal p will lie at the local maximum of this cubic. Since $y' = 3t^2 - 24t + 21 = 3(t-1)(t-7)$, the local maximum occurs at $t = 1$, so the local maximum is $1^3 - 12 \cdot 1^2 + 21 \cdot 1 = \boxed{10}$ (this can be achieved by letting $(x, y, z) = (1, 1, 10)$).

7. **Answer: $\frac{11}{256}$**

Call the three numbers x, y, and z. By symmetry, we need only consider the case $2 \geq x \geq y \geq z \geq 0$. Plotted in 3D, the values of (x, y, z) satisfying these inequalities form a triangular pyramid with a leg-2 right isosceles triangle as its base and a height of 2, with a volume of $2 \cdot 2 \cdot \frac{1}{2} \cdot 2 \cdot \frac{1}{3} = \frac{4}{3}$. We now need the volume of the portion of the pyramid satisfying $x - z \leq \frac{1}{4}$. The equation $z = x - \frac{1}{4}$ is a plane which slices off a skew triangular prism along with a small triangular pyramid at one edge of the large triangular pyramid. The prism has a leg-$\frac{1}{4}$ right isosceles triangle as its base and a height of $\frac{7}{4}$, so has volume $\frac{1}{4} \cdot \frac{1}{4} \cdot \frac{1}{2} \cdot \frac{7}{4} = \frac{7}{2^7}$. The small triangular pyramid also has a leg-$\frac{1}{4}$ right isosceles triangle as its base and a height of $\frac{1}{4}$, so has volume $\frac{1}{4} \cdot \frac{1}{4} \cdot \frac{1}{2} \cdot \frac{1}{4} \cdot \frac{1}{3} = \frac{1}{3 \cdot 2^7}$. Then our probability is $\left(\frac{7}{2^7} + \frac{1}{3 \cdot 2^7}\right) / \left(\frac{4}{3}\right) = 11/256$.

8. **Answer: $\frac{1}{7}$**

Let x be the probability that Frank reaches the cheese before the mousetrap, starting from the top left. Let y be the probability that Frank reaches the cheese before the mousetrap, starting from the top right or the bottom left (which are symmetric).

After 2 moves from the top left there is $\frac{1}{3}$ chance that Frank returns to the top left corner, there is $\frac{1}{3}$ chance that Frank reaches the mousetrap, and there is $\frac{1}{3}$ chance that Frank reaches the top right or bottom left corners. This gives us the relation

$$x = \frac{1}{3}x + \frac{1}{3}0 + \frac{1}{3}y.$$

After 2 moves from the top right corner there is $\frac{1}{3}$ chance that Frank returns to the top right corner, $\frac{1}{3}$ chance that Frank reaches the mousetrap, $\frac{1}{6}$ chance that Frank reaches the top left corner, and $\frac{1}{6}$ chance that Frank reaches the cheese. This gives the relation

$$y = \frac{1}{3}y + \frac{1}{3}0 + \frac{1}{6}x + \frac{1}{6}.$$

Now we have a system of linear of equations and we solve, obtaining $x = \frac{1}{7}$.

9. **Answer: $\sqrt{x} + \sqrt{y} = 1$ or equivalent form**

The limiting curve is the boundary of a region given by the union of all line segments connecting $(q, 0)$ and $(0, 1-q)$ for all numbers $0 \leq q \leq 1$. Such a line segment has equation $\frac{x}{q} + \frac{y}{1-q} = 1$. Thus a point (x_0, y_0) is in that region if and only if the equation $\frac{x}{q} + \frac{y}{1-q} = 1$, $(1-q)x + qy = q(1-q)$ has a solution in $0 \leq q \leq 1$. Let $F(q) = (1-q)x + qy - q(1-q) = q^2 - (1 + x - y)q + x$. Note that $F(0) = x \geq 0$ and $F(1) = y \geq 0$, and the minimum of F at $\frac{1+x-y}{2}$ is always between 0 and 1. So F has a root in $[0, 1]$ if and only if $F(\frac{1+x-y}{2}) = -\frac{(1+x-y)^2}{4} + x \leq 0$. So $4x \leq (1 + x - y)^2$, $2\sqrt{x} \leq 1 + x - y$, $y \leq 1 - 2\sqrt{x} + x = (1 - \sqrt{x})^2$, $\sqrt{y} \leq 1 - \sqrt{x}$, and finally we have $\sqrt{x} + \sqrt{y} \leq 1$.

10. **Answer: $2011^2 - 2011 + 2 = 4042112$**

Let $f(n)$ denote the maximum number of regions into which n circles can partition the plane. We will show that $f(n)$ is a quadratic polynomial in n. Indeed, let A be a planar arrangement of n circles. Note that A is a graph: Each intersection point is a vertex, and the arcs connecting them are edges. Having recognized this, we can apply Euler's theorem, $V - E + F = 2$, to find the number of regions (i.e., F). It is easy to see that an arrangement with the maximum number of vertices is optimal. The maximum number of vertices is $V = 2\binom{n}{2} = n(n-1)$, since each circle can intersect each other circle in at most two vertices. In this optimal arrangement, each circle contains $2(n-1)$ vertices and the same number of edges; thus, the total number of edges is $E = 2n(n-1)$. Thus, the desired quantity is $f(n) = E - V + 2 = n^2 - n + 2$, so our answer is $2011^2 - 2011 + 2 = 4042112$.

Alternative Solution: As before, we apply Euler's theorem for planar graphs. Given that circles are defined by quadratic polynomials, it is clear that V and E are each quadratic in n. In particular, Euler's theorem implies that F is quadratic in n. Moreover, it is easy to check that $f(1) = 2$, $f(2) = 4$, and $f(3) = 8$. Interpolating gives $f(n) = n^2 - n + 1$, as in the first solution.

11. **Answer:** $\frac{1}{4}$

If we consider the triangle ABC with side length $AB = x + y$, $BC = y + z$, $CA = z + x$, the equation becomes

$$\frac{|ABC|^2}{AB^2 \cdot BC^2} = \frac{\sin^2 B}{4} \leq \boxed{\frac{1}{4}}.$$

12. **Answer:** $x^2 - 4y - 4 = 0$

Let $O = (0, 0, 1)$ be the center of the sphere. For a point $X = (x, y, 0)$ on the boundary of the projection, the angle $\angle XPO$ is constant as X varies, since it is just the angle between OP and any tangent from P to the sphere. Considering the case when $X = (0, -1, 0)$, we can see that $\angle XPO = 45°$. Writing this in terms of the dot product, one has $(\overrightarrow{PO} \cdot \overrightarrow{PX})^2 = \frac{1}{2}|\overrightarrow{PO}|^2|\overrightarrow{PX}|^2$, which is equivalent to $((0, 1, -1) \cdot (x, y + 1, -2))^2 = \frac{1}{2}|(0, 1, -1)|^2|(x, y + 1, -2)|^2$, or $(y + 3)^2 = x^2 + (y + 1)^2 + 4$. The answer is $x^2 - 4y - 4 = 0$.

13. **Answer:** 2^{2011}

Define $z_k = x_k + iy_k$. Then the equations are equivalent to $z_{k+1} = {z_k}^2 - 2$, $z_{2012} = z_1$. Let α be a solution of $z_1 = \alpha + \alpha^{-1}$ (which always has two distinct solutions unless $z_1 = 2$ or -2). Then one can check by induction that $z_k = \alpha^{2^{k-1}} + \alpha^{-2^{k-1}}$. Since one has $z_{2012} = z_1$, $\alpha^{2^{2011}} + \alpha^{-2^{2011}} = \alpha + \alpha^{-1}$.

Set $N = 2^{2011}$ and rewrite the above as $\alpha^{2N} + 1 = \alpha^{N-1} + \alpha^{N+1}$, or $(\alpha^{N+1} - 1)(\alpha^{N-1} - 1) = 0$. Since N is even, $N + 1$ and $N - 1$ are relatively prime. So the equations $X^{N+1} = 1$ and $X^{N-1} = 1$ have only the root 1 in common. Therefore there are $(N + 1) + (N - 1) - 1 = 2N - 1$ possibilities for α. Meanwhile, any one value of $z_1 = \alpha + \alpha^{-1}$ corresponds to two choices of α except when $\alpha = 1$ or -1. So our $2N - 2$ choices of $\alpha \neq 1$ together give $N - 1$ different solutions for z_1, and $\alpha = 1$ give a single solution $z = 2$. The answer is $N = 2^{2011}$.

14. **Answer:** $\frac{\pi \ln(2)}{8}$

Let I denote the integral we wish to compute. The function $f(x) = \dfrac{\ln(x+1)}{x^2+1}$ does not have an elementary antiderivative. We will use Taylor series to compute I. We can find the Taylor series for the function $\dfrac{\ln(x+1)}{x^2+1}$ using the following formulas:

$$\ln(x+1) = x - \frac{x^2}{2} + \frac{x^3}{3} - \dots$$
$$\frac{1}{1+x^2} = 1 - x^2 + x^4 - \dots$$

These formulas aren't good everywhere, but they do hold in $(0, 1)$. We have

$$\begin{aligned} f(x) &= \left(x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots\right)\left(1 - x^2 + x^4 - x^6 + \dots\right) \\ &= x + \left(-\frac{1}{2}\right)x^2 + \left(\frac{1}{3} - 1\right)x^3 + \left(-\frac{1}{4} + \frac{1}{2}\right)x^4 + \left(\frac{1}{5} - \frac{1}{3} + 1\right)x^5 + \dots \end{aligned}$$

In particular, an antiderivative is given by

$$F(x) = \frac{1}{2}x^2 + \frac{1}{3}\left(-\frac{1}{2}\right)x^3 + \frac{1}{4}\left(\frac{1}{3} - 1\right)x^4 + \frac{1}{5}\left(-\frac{1}{4} + \frac{1}{2}\right)x^5 + \frac{1}{6}\left(\frac{1}{5} - \frac{1}{3} + 1\right)x^6 + \dots$$

The definite integral I is given by $F(1)$, i.e., the sum

$$I = \frac{1}{2} + \frac{1}{3}\left(-\frac{1}{2}\right) + \frac{1}{4}\left(\frac{1}{3} - 1\right) + \frac{1}{5}\left(-\frac{1}{4} + \frac{1}{2}\right) + \frac{1}{6}\left(\frac{1}{5} - \frac{1}{3} + 1\right) + \dots$$

Now we use the facts that

$$1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\ldots=\frac{\pi}{4}$$
$$1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\ldots=\ln(2),$$

from the Taylor series for $\tan^{-1}(x)$ and $\ln(x+1)$ respectively. Notice that in the above sum, every number of the form $\frac{1}{r\cdot s}$, where r is even and s is odd, occurs exactly once, with a positive sign if $r+s\equiv 3 \pmod 4$ and a negative sign if $1 \pmod 4$. Therefore, it is clear that

$$\begin{aligned}I&=\left(1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\ldots\right)\left(\frac{1}{2}-\frac{1}{4}+\frac{1}{6}-\frac{1}{8}+\ldots\right)\\&=\frac{\pi}{4}\cdot\frac{1}{2}\left(1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\ldots\right)\\&=\frac{\pi\ln(2)}{8}.\end{aligned}$$

15. **Answer:** $\mathbf{\frac{1}{2}}$

Note that both $\gcd(a,b-1)$ and $\gcd(a-1,b)$ divide $a+b-1$. Also they are relatively prime, since $\gcd(a,b-1)\mid a$ and $\gcd(a-1,b)\mid a-1$. So their product is less than or equal to $a+b-1$, and therefore by the AM-GM inequality we have

$$\frac{1}{\gcd(a,b-1)}+\frac{1}{\gcd(a-1,b)}\geq 2\sqrt{\frac{1}{\gcd(a,b-1)\cdot\gcd(a-1,b)}}\geq\frac{2}{\sqrt{a+b-1}}.$$

Thus $\alpha=\frac{1}{2}$ and $m=2$ suffice. To show that there is no such m for smaller α, let $b=(a-1)^2$. Then $\gcd(a,b-1)=a$ and $\gcd(a-1,b)=a-1$, so

$$\left(\frac{1}{\gcd(a,b-1)}+\frac{1}{\gcd(a-1,b)}\right)(a+b)^{\alpha}=\frac{(2a-1)(a^2-a+1)^{\alpha}}{a(a-1)}$$

and the limit when a goes to ∞ is zero if $\alpha<\frac{1}{2}$.