1. **Answer: 2100 Martian dollars**

   This problem should be approached through brute force since there are few possible combinations. The class should rent three shuttles of type A, one shuttle of type B, and one shuttle of type C.

2. **Answer: $\frac{50}{23}$**

   Add the rates all three vicious animals together, $\frac{1}{4}t = \frac{1}{5}t + \frac{1}{100}t = 1$ and hence $t = \frac{50}{23}$ hours.

3. **Answer: 15 kg**

   The mass of the bucket when empty is $29-(29-18.5)*2 = 8$ kg. When filled to one third of maximum capacity, it will have mass $8+(29-8)/2 = 15$ kg.

4. **Answer: 32**

   We can show trigonometrically that square inscribed in the circle maximizes the area, so we look for the square of largest size. This square has diagonal 8. Therefore, the square has area $\left(\frac{8}{\sqrt{2}}\right)^2 = 32$.

5. **Answer: 3**

   You can check that for 0 quarters, there must be 3 dimes, 8 nickels, and 4 pennies. For 1 quarter, there must be 3 dimes, 2 nickels, and 9 pennies. There are no 15 coin combinations that yield 74 cents. Hence, there must be 3 dimes.

6. **Answer: 14**

   Using $f(n) = \frac{n(n+1)}{2}$, we notice that $f(13) = 91$ and $f(14) = 105$. Hence, the answer is 14.

7. **Answer: $\max\{n, m\}$**

   The points of intersection are the zeros of the difference between the polynomials. This difference is a polynomial of degree at most $\max\{n, m\}$.

8. **Answer: $2^{98} \times 5$**

$$\begin{aligned} a_n &= a_{n-1} + 2a_{n-2} \\ a_n + a_{n-1} &= 2(a_{n-1} + a_{n-2}) \\ &= 2^{n-2}(a_1 + a_2). \end{aligned}$$

   So $a_{100} + a_{99} = 2^{98} \times 5$.

9. **Answer: $\frac{861}{80}$**

   Split B into two series C and D where the terms of C are $c_j = a_j^2 = \frac{49}{16}\left(\frac{4}{9}\right)^{j-1}$ and the terms of D are $d_j = a_j = \frac{7}{4}\left(\frac{2}{3}\right)^{j-1}$. Since both C and D are geometric series with ratios less than 1, the sum of their terms yields $\frac{49/16}{1-4/9} = \frac{441}{80}$ and $\frac{7/4}{1-2/3} = \frac{21}{4}$. Therefore, the sum of the terms in B equals $\frac{441}{80} + \frac{21}{4} = \frac{861}{80}$.

10. **Answer: $\frac{3}{4}$**

   Setting $x = 2$, we find that $F(2) + F\left(\frac{1}{2}\right) = 3$. Now take $x = \frac{1}{2}$, to get that $F\left(\frac{1}{2}\right) + F(-1) = \frac{3}{2}$. Finally, setting $x = -1$, we get that $F(-1) + F(2) = 0$. Then we find that

$$\begin{aligned} F(2) &= 3 - F\left(\frac{1}{2}\right) = 3 - \left(\frac{3}{2} - F(-1)\right) = \frac{3}{2} + F(-1) = \frac{3}{2} - F(2) \\ &\Rightarrow F(2) = \frac{3}{4}. \end{aligned}$$

   Alternate Solution: We can explicitly solve for $F(x)$ and then plug in $x = 2$. Notice that for $x \neq 0, 1$, $F(x) + F\left(\frac{x-1}{x}\right) = 1 + x$ so

$$F\left(\frac{x-1}{x}\right) + F\left(\frac{1}{1-x}\right) = 1 + \frac{x-1}{x} \text{ and } F\left(\frac{1}{1-x}\right) + F(x) = 1 + \frac{1}{1-x}.$$

Thus

$$\begin{aligned}2F(x) &= F(x) + F\left(\frac{x-1}{x}\right) - F\left(\frac{x-1}{x}\right) - F\left(\frac{1}{1-x}\right) + F\left(\frac{1}{1-x}\right) + F(x)\\ &= 1 + x - \left(1 + \frac{x-1}{x}\right) + 1 + \frac{1}{1-x}\\ &= 1 + x + \frac{1-x}{x} + \frac{1}{1-x}.\end{aligned}$$

It follows that $F(x) = \frac{1}{2}\left(1 + x + \frac{1-x}{x} + \frac{1}{1-x}\right)$ and the result follows by taking $x = 2$.

11. **Answer: 1818.2 m**

    The total time it takes for Stephen and Tal to meet is the total distance in meters (2000 m) divided by their relative velocity ($1.2 + 1 = 2.2$ m/s). Thus, the total time is $\frac{2000}{2.2}$. The dog runs at 2 m/s for the entire time, so it runs $2\frac{2000}{2.2} = \frac{20000}{11} \approx 1818.2$ m.

12. **Answer: 57.2 minutes**

    When in use, the laptop depletes the battery at a rate of $\frac{1}{286}$ of the total capacity per minute. It charges at a rate of $\frac{1}{52}$ of the total capacity per minute. Thus, when the laptop is in use and charging, the battery increases at a rate of $\frac{1}{52} - \frac{1}{286} = \frac{9}{572}$ per minute. Hence, it takes $\frac{572}{9} * .9 = 57.2$ minutes to fully charge the battery under the stated conditions.

13. **Answer: $\frac{1}{3}$**

    It is easy to show that it is greater than or equal to 1/3, because

    $$(a + b + c) + (c + d + e) + (e + f + g) \geq a + b + c + d + e + f + g = 1.$$

    The minimum is achieved by

    $$(a, b, c, d, e, f, g) = (1/3, 0, 0, 1/3, 0, 0, 1/3).$$

14. **Answer: 189**

    Because 20 and 11 are relatively prime, the largest number that cannot be expressed as $am + bn$ for positive integers a and b is $mn - m - n$, so the answer is $20 \times 11 - 20 - 11 = 189$.

15. **Answer: $\frac{5n^2+6n+1}{6n^2}$**

    Notice that $\sum_{i=0}^{n}\left(\frac{i}{n}\right)^2 = \frac{1}{n^2}\sum_{i=0}^{n} i^2 = \frac{(n+1)(2n+1)}{6n}$. Also note that $\sum_{i=0}^{n}\frac{i}{n} = \frac{1}{n}\sum_{i=0}^{n} i = \frac{n+1}{2n}$. Therefore, the answer is $\frac{5n^2+6n+1}{6n^2}$.

16. **Answer: 63**

    This amounts to determining, for a given numerator, how many elements in S are relatively prime to the numerator. If we let $f(n)$ be the number of positive integers relatively prime to n and less than or equal to 10, it is obvious that $f(1) = 10$, $f(2) = f(4) = f(8) = 5$, $f(3) = f(9) = 7$, $f(5) = 8$, $f(7) = 9$, $f(6) = 3$, and $f(10) = 4$. Therefore, the answer is $10 + 3 \cdot 5 + 2 \cdot 7 + 8 + 9 + 3 + 4 = 63$.

17. **Answer: $2^2, 3^2, 5^2$**

    For $b = 0$ one has $2^a + 1 = c^2$, $2^a = (c+1)(c-1)$. Thus both $c + 1$ and $c - 1$ should be powers of 2. The only possibility is $c = 3$, which gives a solution $2^3 + 3^0 = 9 = 3^2$.
    For $b \geq 1$, $2^a + 3^b$ is not divisible by 3, so it should be $\equiv 1 (\text{mod } 3)$. This requires a to be even. Let $a = 2d$, then $3^b = c^2 - 2^{2d} = (c + 2^d)(c - 2^d)$. Let $c + 2^d = 3^p$ and $c - 2^d = 3^q$. Eliminating c, one has $2^{d+1} = 3^p - 3^q$. For $q \geq 1$ the right-hand side is divisible by 3, so $q = 0$. From what we know, there are only two solutions $(d, p) = (0, 1), (2, 2)$. These solutions give $2^0 + 3^1 = 4 = 2^2$ and $2^4 + 3^2 = 25 = 5^2$ respectively.

18. **Answer: 146**

    Let f_n be the number of ways to jump from zero to n, ignoring for the time being jumping backwards.. We have $f_0 = 1$, $f_1 = 1$, and $f_n = f_{n-1} + f_{n-2}$ when $n \geq 2$. Therefore, we have that $f_2 = 2$, $f_3 = 3$, $f_4 = 5$, $f_5 = 8$, $f_6 = 13$, and $f_7 = 21$. Note that we can describe the frog's jumping as jumping forward n numbers, jumping backward 1 number, and jumping forward $8 - n$ numbers. Therefore, the desired answer is simply $\sum_{i=1}^{6} f_i f_{8-i} = 146$.

19. **Answer: 872**

    Let a_n denote the number of sparse numbers with no more than n binary digits. In particular, for numbers with less than n binary digits after removing leading zeroes, append leading zeroes so all numbers have n binary digits when including sufficiently many leading zeroes. We have that $a_0 = 1$, $a_1 = 2$, and $a_2 = 3$ since for these lengths, either zero digits are 1 or one digit is 1. We claim that the recurrence $a = a_{n-1} + a_{n-3}$ holds for $n \geq 3$. We split this analysis into two cases; numbers where the nth binary digit is 0 or 1. When the nth binary digit is zero, we can remove that zero to get a valid number with $n - 1$ binary digits. When the nth binary digit is one, it is known that the $(n-1)$th and $(n-2)$th digits are both zero, so we can truncate those to get a valid number with $n - 3$ binary digits. Therefore, the recurrence holds. With the given initial conditions, $a_{17} = 872$.

20. **Answer: 1**

    Let $P = (a, 0)$. Note that $\angle MPN$ is inscribed in the circle defined by points M, P, and N, and that it intercepts MN. Since MN is fixed, it follows that maximizing the measure of $\angle MPN$ is equivalent to minimizing the size of the circle defined by M, P, and N. Since P must be on the x-axis, we therefore want this circle to be tangent to the x-axis. Since the center of this circle must lie on the perpendicular bisector of MN, which is the line $y = 3 - x$, the center of the circle has to be of the form $(a, 3 - a)$, so a has to satisfy $(a+1)^2 + (1-a)^2 = (a-3)^2$. Solving this equation gives $a = 1$ or $a = -7$. Clearly choosing a = 1 gives a smaller circle, so our answer is 1.

21. **Answer: 60**

    There are 12 vertices, each with 5 neighbors. Any vertex and any of its neighbors can be rotated to any other vertex-neighbor pair in exactly one way. There are $5 \cdot 12 = 60$ vertex-neighbor pairs.

22. **Answer: $\boldsymbol{x = e^{2011\pi/5}}$**

    Set $y = \ln x$, and observe that

    $$2\cos(3y)\sin(2y) = \sin(3y + 2y) - \sin(3y - 2y) = \sin(5y) - \sin(y),$$

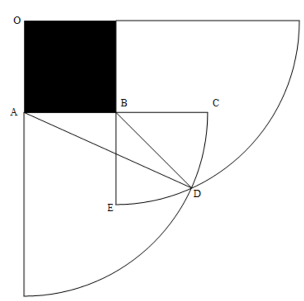
    so that the equation in question is simply

    $$\sin(5y) = 0.$$

    The solutions are therefore

    $$\ln x = y = \frac{n\pi}{5} \implies x = e^{n\pi/5} \quad \text{for all } n \in \mathbb{N}$$

23. **Answer: $\boldsymbol{\frac{79\pi}{2} + \frac{3}{2} - 5x}$**



Label the top left corner of the square as the origin O. By keeping the leash straight, the ant can travel through $\frac{3}{4}$ of a circle of radius 6 ($A_1 = \frac{3}{4} \times 36\pi = 27\pi$). The ant can also bend the leash around the two nearest corners of the square to where it is leashed ($A_2 = 2 \times \frac{1}{4} \times 25\pi = \frac{25}{2}\pi$). However, this double counts the area enclosed by BCDE, which is equal to two times the area of BCD. To calculate the latter, notice that ACD is the sector of the circle centered at A with radius 5. We can calculate the coordinates of D usign the two equations $y = -x$ (from the symmetry) and $x^2 + (y+1)^2 = 5^2$ which yields $D = (4, -4)$. Since $A = (0, -1)$, the angle of sector ACD is $\arctan\left(\frac{3}{4}\right) = x$. The area of triangle ABD equals $\frac{3}{2}$ (base times height) so BCD has area $5x - \frac{3}{2}$ and BCDE has area $10x - 3$. Hence, the total area is $A_1 + A_2 - \left(5x - \frac{3}{2}\right) = \frac{79\pi}{2} + \frac{3}{2} - 5x$.

24. **Answer: 7**

Using Menelaus's Theorem on $\triangle ABD$ with collinear points F, X, C and the provided ratios gives $DX/XA = 4/3$. Using Menelaus's Theorem on $\triangle ADC$ with collinear points B, Y, E gives $AY/YD = 6$. We conclude that AX, XY, YD are in length ratio $3:3:1$. By symmetry, this also applies to the segments CZ, ZX, XF and BY, YZ, ZE. Repeatedly using the fact that the area ratio of two triangles of equal height is the ratio of their bases, we find $[ABC] = (3/2)[ADC] = (3/2)(7/3)[XYC] = (3/2)(7/3)(2)[XYZ] = 7[XYZ]$, or $[ABC]/[XYZ] = 7$.

Alternate Solution

Stretching the triangle will preserve ratios between lengths and ratios between areas, so we may assume that $\triangle ABC$ is equilateral with side length 3. We now use mass points to find the length of XY. Assign a mass of 1 to A. In order to have X be the fulcrum of $\triangle ABC$, C have mass 2 and B must have mass 4. Hence, $BX : XE = 4 : 3$ and $AX : XD = 6 : 1$, the latter of which also equals $BY : YE$ by symmetry. Hence, $XY = \frac{3}{7}BE$. To find BE, we apply the Law of Cosines to $\triangle CBE$ to get that $BE^2 = 1^2 + 3^2 - 2 \cdot 1 \cdot 3 \cdot \cos 60^\circ = 7 \implies XY = \frac{3\sqrt{7}}{7}$. Since $\triangle XYZ$ must be equilateral by symmetry, the desired ratio equals $(\frac{AB}{XY})^2 = 7$.

25. **Answer: $\frac{\sqrt{5}+1}{4}$**

One has $ab \leq \frac{t}{2}a^2 + \frac{1}{2t}b^2$, $bc \leq \frac{1}{2}b^2 + \frac{1}{2}c^2$, and $cd \leq \frac{1}{2t}c^2 + \frac{t}{2}d^2$ by AM-GM. If we can set t such that $\frac{t}{2} = \frac{1}{2t} + \frac{1}{2}$, it can be proved that $\frac{ab+bc+cd}{a^2+b^2+c^2+d^2} \leq \frac{\frac{t}{2}(a^2+b^2+c^2+d^2)}{a^2+b^2+c^2+d^2} = \frac{t}{2}$, and this is maximal because we can set a, b, c, d so that the equality holds in every inequality we used. Solving this equation, we get $t = \frac{1+\sqrt{5}}{2}$, so the maximum is $\frac{t}{2} = \frac{\sqrt{5}+1}{4}$.