1. **Answer: \$88**

Let x be the amount of money he invests each year. We make the following table about the amount of money he has:

Year	Money on Jan 1	Money on Dec 31
8	x	$2x$
9	$2x+x$	2^2x+2x
10	$2^2x+2x+x$	2^3x+2^2x+2x
...	...	...
17	$2^9x+2^8x+\ldots+2x+x$	$2^{10}x+2^9x+\ldots+2x$

Sammy needs $4\times\$45000=\$180000$. Then $180000=2^{10}x+2^9x+\ldots+2x=2(2^9+2^8+\ldots+2+1)x=2(2^{10}-1)x=2046x\Rightarrow x=180000/2046\simeq 87.97$ so the least integer amount of money he needs to invest is \$88.

2. **Answer: $\left(-\frac{3}{5},\frac{4}{5}\right)$**

From the first equation, we get that $y^2=1-x^2$. Plugging this into the second one, we are left with

$$\begin{aligned}2x^2\pm 2x\sqrt{1-x^2}+1-x^2-2x\mp 2\sqrt{1-x^2}=0&\Rightarrow (x-1)^2=\mp 2\sqrt{1-x^2}(x-1)\\&\Rightarrow x-1=\mp 2\sqrt{1-x^2}\text{ assuming } x\neq 1\\&\Rightarrow x^2-2x+1=4-4x^2\Rightarrow 5x^2-2x-3=0.\end{aligned}$$

The quadratic formula yields that $x=\frac{2\pm 8}{10}=1,-\frac{3}{5}$ (we said that $x\neq 1$ above but we see that it is still valid). If $x=1$, the first equation forces $y=0$ and we easily see that this solves the second equation. If $x=-\frac{3}{5}$, then clearly y must be positive or else the second equation will sum five positive terms. Therefore $y=\sqrt{1-\frac{9}{25}}=\sqrt{\frac{16}{25}}=\frac{4}{5}$. Hence the other point is $\left(-\frac{3}{5},\frac{4}{5}\right)$.

3. **Answer: $x=-1,0,2$**

There are four intervals to consider, each with their own restrictions. Consider the case in which $x>\sqrt{2}$. Then the equation becomes $(x-1)(x^2-2)-2=x(x-2)(x+1)=0$. Thus, $x=2$ is the only rational root for $x>\sqrt{2}$. Consider the case in which $-\sqrt{2}<x<1$. Then the equation becomes $(x-1)(x^2-2)-2=x(x-2)(x+1)=0$. Thus, $x=0$ and $x=-1$ are the rational roots for $-\sqrt{2}<x<1$. Consider the case in which $x<-\sqrt{2}$ or the case in which $1<x<\sqrt{2}$. In these cases, the equation becomes $(1-x)(x^2-2)-2=-x^3+x^2+2x-4$. By the rational root theorem, the rational roots of this polynomial can only be $\pm 4,\pm 2,\pm 1$ and a quick check shows that none of these are roots, so this polynomial has no rational roots.

4. **Answer: $\frac{9}{4}$**

First notice that the polynomial

$$g(x)=x^4\left(\frac{1}{x^4}+\frac{3}{x^3}+\frac{3}{x}+2\right)=2x^4+3x^3+3x+1$$

is a polynomial with roots $\frac{1}{r},\frac{1}{s},\frac{1}{t},\frac{1}{u}$. Therefore, it is sufficient to find the sum of the squares of the roots of $g(x)$, which we will denote as r_1 through r_4. Now, note that

$$r_1^2+r_2^2+r_3^2+r_4^2=(r_1+r_2+r_3+r_4)^2-(r_1r_2+r_1r_3+r_1r_4+r_2r_3+r_2r_4+r_3r_4)=(-\frac{a_3}{a_4})^2-\frac{a_2}{a_4}$$

by Vieta's Theorem, where a_n denotes the coefficient of x^n in $g(x)$. Plugging in values, we get that our answer is $(-\frac{3}{2})^2-0=\frac{9}{4}$.

5. **Answer: $\frac{33}{2}$**

   Note that $\frac{7n+32}{n(n+2)} = \frac{16}{n} - \frac{9}{n+2}$ so that

$$\begin{aligned}\sum_{n=1}^{\infty} \frac{(7n+32)}{n(n+2)} \frac{3^n}{4^n} &= \sum_{n=1}^{\infty} \frac{16}{n}\frac{3^n}{4^n} - \sum_{n=1}^{\infty} \frac{9}{n+2}\frac{3^n}{4^n} \\ &= \sum_{n=1}^{\infty} \frac{16}{n}\frac{3^n}{4^n} - \sum_{n=1}^{\infty} \frac{16}{n+2}\frac{3^{n+2}}{4^{n+2}} \\ &= \sum_{n=1}^{\infty} \frac{16}{n}\frac{3^n}{4^n} - \sum_{n=3}^{\infty} \frac{16}{n}\frac{3^n}{4^n} \\ &= \frac{16}{1}\frac{3}{4} + \frac{16}{2}\frac{9}{16} = \frac{33}{2}.\end{aligned}$$

6. **Answer: $(-3^{1005}-1)x + (-2\cdot 3^{1005} - 1)$**

   The standard method is to use the third root of unity ω, $\omega^2+\omega+1=0$. Let $(x+2)^{2011}-(x+1)^{2011} = (x^2+x+1)Q(x)+ax+b$ and substitute $x=\omega$. Then $a\omega+b = (\omega+2)^{2011}-(\omega+1)^{2011}$. Note that $\omega+2$ has size $\sqrt{3}$ and argument $\pi/6$, so $(\omega+2)^6 = -3^3$. Also $\omega+1$ has magnitude 1 and argument $\pi/3$, so $(\omega+1)^6 = 1$. Using this and $2011 = 6\cdot 335+1$, we get that $a\omega+b = (-3^{1005}-1)\omega + (-2\cdot 3^{1005}-1)$.

   Another solution is to note that $(x+2)^2 \equiv x^2+4x+4 \equiv -3x^2 \pmod{x^2+x+1}$ and $(x+1)^2 \equiv x^2+2x+1 \equiv x \pmod{x^2+x+1}$. Then we have $x^3 \equiv 1 \pmod{x^2+x+1}$ and we can proceed by using periodicity.

7. **Answer: $\frac{8045}{2012}$**

   Let $y_1, y_2, \cdots, y_{2010}$ be the 2010 numbers distinct from x. Then $y_1+y_2+\cdots+y_{2010} = 2012-x$ and $\frac{1}{y_1}+\frac{1}{y_2}+\cdots+\frac{1}{y_{2010}} = 2012-\frac{1}{x}$. Applying the Cauchy-Schwarz inequality gives

$$\left(\sum_{i=1}^{2010} y_i\right)\left(\sum_{i=1}^{2010} \frac{1}{y_i}\right) = (2012-x)(2012-\frac{1}{x}) \geq 2010^2$$

   so $2012^2 - 2012(x+x^{-1}) + 1 - 2010^2 \geq 0$, $x+x^{-1} \leq 8045/2012$.

8. **Answer: $2 - \frac{1}{2^{2010}}$**

   We analyze $Q(x) = P(2x)-P(x)$. One can observe that $Q(x)-1$ has the powers of 2 starting from $1, 2, 4, \cdots$, up to 2^{2010} as roots. Since Q has degree 2011, $Q(x)-1 = A(x-1)(x-2)\cdots(x-2^{2010})$ for some A. Meanwhile $Q(0) = P(0)-P(0) = 0$, so

$$Q(0)-1 = -1 = A(-1)(-2)\cdots(-2^{2010}) = -2^{(2010\cdot 2011)/2}A.$$

   Therefore $A = 2^{-(1005\cdot 2011)}$. Finally, note that the coefficient of x is same for P and $Q-1$, so it equals $A(-2^0)(-2^1)\cdots(-2^{2010})((-2^0)+(-2^{-1})+\cdots+(-2^{-2010})) = \frac{A\cdot 2^{1005\cdot 2011}(2^{2011}-1)}{2^{2010}} = 2-\frac{1}{2^{2010}}$.

9. **Answer: -665**

   Since the equation

$$P_k(x) = P_k(x-1) + x^k$$

   has all integers ≥ 2 as roots, the equation is an identity, so it holds for all x. Now we can substitute $x = -1, -2, -3, -4, \cdots$ to prove

$$P_k(-n) = -\sum_{i=1}^{n-1} (-i)^k$$

   so $P_7(-3) + P_6(-4) = -(-1)^6 - (-2)^6 - (-3)^6 - (-1)^7 - (-2)^7 = -665$.

10. **Answer: 10**

Note that if r is a root of P then r^2 is also a root. Therefore $r, r^2, r^{2^2}, r^{2^3}, \cdots$, are all roots of P. Since P has a finite number of roots, two of these roots should be equal. Therefore, either $r = 0$ or $r^N = 1$ for some $N > 0$.

If all roots are equal to 0 or 1, then P is of the form $ax^b(x-1)^{(4-b)}$ for $b = 0, ..., 4$.

Now suppose this is not the case. For such a polynomial, let q denote the largest integer such that $r = e^{2\pi i \cdot p/q}$ is a root for some integer p coprime to q. We claim that the only suitable $q > 1$ are $q = 3$ and $q = 5$.

First note that if r is a root then one of $\sqrt{r}$ or $-\sqrt{r}$ is also a root. So if q is even, then one of $e^{2\pi i \cdot p/2q}$ or $e^{2\pi i \cdot p+q/2q}$ should also be root of p, and both p/q and $(p+q)/2q$ are irreducible fractions. This contradicts the assumption that q is maximal. Therefore q must be odd. Now, if $q > 6$, then $r^{-2}, r^{-1}, r, r^2, r^4$ should be all distinct, so $q \leq 6$. Therefore $q = 5$ or 3.

If $q = 5$, then the value of p is not important as P has the complex fifth roots of unity as its roots, so $P = a(x^4 + x^3 + x^2 + x + 1)$. If $q = 3$, then P is divisible by $x^2 + x + 1$. In this case we let $P(x) = a(x^2 + x + 1)Q(x)$ and repeating the same reasoning we can show that $Q(x) = x^2 + x + 1$ or $Q(x)$ is of form $x^b(x-1)^{2-b}$.

Finally, we can show that exactly one member of all 10 resulting families of polynomials fits the desired criteria. Let $P(x) = a(x-r)(x-s)(x-t)(x-u)$. Then, $P(x)P(-x) = a^2(x^2-r^2)(x^2-s^2)(x^2-t^2)(x^2-u^2)$. We now claim that r^2, s^2, t^2, and u^2 equal r, s, t, and u in some order. We can prove this noting that the mapping $f(x) = x^2$ maps 0 and 1 to themselves and maps the third and fifth roots of unity to another distinct third or fifth root of unity, respectively. Hence, for these polynomials, $P(x)P(-x) = a^2(x^2-r)(x^2-s)(x^2-t)(x^2-u) = aP(x^2)$, so there exist exactly 10 polynomials that fit the desired criteria, namely the ones from the above 10 families with $a = 1$.