Definitions

A graph is a collection of points (vertices) connected by line segments (edges). In this test, all graphs will be simple – any two vertices will be connected by at most one edge – and connected – you can get from any vertex to any other by following edges.

A simple graph with 7 vertices and 11 edges.

An edge n-coloring of a graph $G$ is an assignment of one of $n$ colors to each edge of $G$.

A 2-coloring of the earlier graph

A complete graph is one in which any two vertices are connected by an edge.

1. a. Draw a simple connected graph with 8 vertices and 7 edges, and 3-color its edges.
   b. Draw a complete graph on 5 vertices, and 2-color its edges so that it does not contain a red triangle or a blue triangle (3 vertices, the edges between which are all red or all blue).

We will use $K_n$ to denote a complete graph on $n$ vertices. A monochromatic $K_n$ is one in which every edge has the same color. Hence, problem 1(b) could have been phrased “Color $K_5$ so that it has no monochromatic $K_3$”.

2. Show that no matter how you 2-color $K_6$, it will contain a monochromatic $K_3$. (Hint: Think about all the edges coming from one vertex).

The Ramsey number $R(k)$ is the least number $n$ such that no matter how you 2-color the edges of $K_n$, there will be a monochromatic $K_k$. In problems 1(b) and 2, you have shown that $R(3) = 6$.

Interestingly, $R(4)$ is a difficult quantity to calculate, and $R(5)$ is still unknown! Since we cannot go much further in this vein, let us try looking at generalizations of Ramsey numbers. Define $R(k, j)$ as the least $n$ such that every red, blue edge 2-coloring of $K_n$ contains either a red $K_k$ or a blue $K_j$. Then $R(n)$ is just $R(n, n)$ under this new definition.

3. a. Show that $R(3, 4) > 8$ by exhibiting a 2-coloring.
   b. Show that $R(4, 3) = 9$ (Hint: Use problem 2.)
4. a. Show that
\[ R(n, m) \leq R(n, m - 1) + R(n - 1, m). \]
(Hint: see hint to problem 2.)
b. Conclude that \( R(n, m) \) is well defined, that is, that it exists for every \( n \) and every \( m \).

From here on, we will explore some interesting properties and generalizations of Ramsey numbers. Each section is independent.

**Bounds on Ramsey Numbers**

5. Color a graph of \( n^2 \) points, laid out in a \( n \times n \) grid, as follows: The edge \((u, v)\) is blue if \( u \) and \( v \) are in the same row, and red otherwise.
   a. Show that any \( K_{n+1} \) in that graph contains at least one red edge and at least one blue edge.
   b. Conclude that \( R(n + 1, n + 1) > n^2 \).

Problem 6 gives us a polynomial lower bound for \( R(n, n) \), and it does so constructively – we know exactly which graph will give a counterexample. Erdős has shown that, if we are willing to be nonconstructive, we can get a much better lower bound:

6. a. Show that if the edges of \( K_m \) are colored red or blue randomly with equal probability (i.e., by flipping a coin for each edge), then the probability that it contains a monochromatic \( K_n \) is at most
\[ \binom{m}{n} \cdot 2^{1 - \binom{n}{2}}. \]
   b. Show that if \( \binom{m}{n} < 2^{\binom{n}{2} - 1} \), that probability is less than 1.
   c. Using the fact that \( \binom{m}{n} < m^n \), show that if \( m = 2^{\frac{n}{2} - \frac{1}{2}} \) then \( \binom{m}{n} < 2^{1 - \binom{n}{2}} \), and conclude that
\[ R(n, n) > 2^{\frac{n}{2} - \frac{1}{2} - \frac{1}{2}}. \]

7. Prove a complementary upper bound: \( R(n, n) \leq 4^n \).

**k-color Ramsey Numbers**

Similar to our definition \( R(n, m) \), we can define \( R(n_1, n_2, n_3, \ldots, n_k) \) to be the least \( m \) such that if \( K_m \) is colored with \( k \) colors, there is some monochromatic \( K_{n_i} \) of color \( c_i \).

8. Prove that \( R(3, 3, 3) \leq 17 \). (In fact, \( R(3, 3, 3) = 17 \), but this is difficult to show.)

9. Show that
\[ R(n_1, \ldots, n_k) \leq R(n_1, n_2, \ldots n_{k-2}, R(n_k, n_k-1)) \]
This gives us the existence of \( R(n_1, \ldots, n_k) \) for all \( \{n_1, \ldots, n_k\} \).

10. Prove that
   a. \( \underbrace{R(3, \ldots, 3)}_{r \text{ 3's}} \leq 3r! \)
b. \[ R(3, \ldots, 3) > 2^r \]

**Infinite Ramsey Numbers**

11. Define \( K_N = (V, E) \), where \( V = \{1, 2, 3, \ldots\} \), and \( E = \{(i, j) : i, j \in V, i < j\} \). This is in some sense an infinite complete graph. Show that if every edge is colored red or blue, there is some infinite subset \( V' \) of \( V \) such that all of the edges between points of \( V' \) are the same color.