1. Compute $\lim_{x\to 0} \dfrac{\tan x - x - \frac{x^3}{3}}{x}$.

2. For how many integers is $\frac{n}{20-n}$ equal to the square of a positive integer.

3. Find all possible solutions $(x_1, x_2, \ldots, x_n)$ to the following equations, where

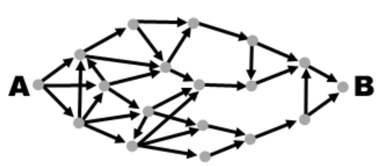
$$\begin{aligned}
x_1 &= \frac{1}{2}\left(x_n + \frac{x_{n-1}^2}{x_n}\right)\\
x_2 &= \frac{1}{2}\left(x_1 + \frac{x_n^2}{x_1}\right)\\
x_3 &= \frac{1}{2}\left(x_2 + \frac{x_1^2}{x_2}\right)\\
x_4 &= \frac{1}{2}\left(x_3 + \frac{x_2^2}{x_3}\right)\\
x_5 &= \frac{1}{2}\left(x_4 + \frac{x_3^2}{x_4}\right)\\
&\vdots\\
x_n &= \frac{1}{2}\left(x_{n-1} + \frac{x_{n-2}^2}{x_{n-1}}\right) = 2010.
\end{aligned}$$

4. There is an economic crisis in Hogwarts. To generate more money for themselves, Fred and George decided to run a magical reaction that will turn spiders into galleons. In a 100L cauldron, the initially put in 50 moles of spiders. In addition, they bewitched their brother Ron to continuously feed in spiders at a rate of 1 mol/min. Assume spiders turn into galleons at a constant at a constant rate of 0.5 mol per minute per liter. Also, assume that the reaction occurs uniformly, that Fred is very assiduous in his stirring job, and there are no other reactions. Find the number of spiders remaining after 1 hour.

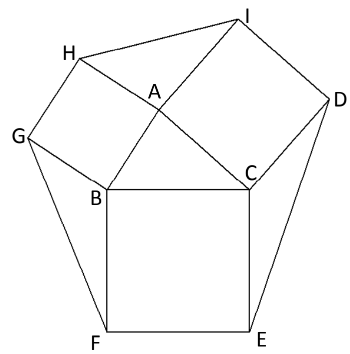
5. Rank the following in decreasing order:

$$A = \frac{1\sqrt{1}+2\sqrt{2}+3\sqrt{3}}{\sqrt{1}+\sqrt{2}+\sqrt{3}}, B = \frac{1^2+2^2+3^2}{1+2+3}, C = \frac{1+2+3}{3}, D = \frac{\sqrt{1}+\sqrt{2}+\sqrt{3}}{\frac{1}{\sqrt{1}}+\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{3}}}.$$

6. How many paths are there from A to B in the directed graph below?



7. $\triangle ABC$ is a triangle with $AB = 5$, $BC = 6$, and $CA = 7$. Squares are drawn on each side, as in the image below. Find the area of hexagon $DEFGHI$.

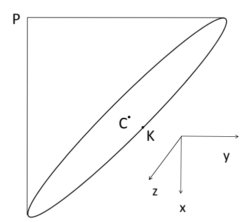


8. Suppose that for an infinitely differentiable function f,

$$\lim_{x\to 0} \frac{f(4x) + af(3x) + bf(2x) + cf(x) + df(0)}{x^4}$$

exists. Find $1000a + 100b + 10c + d$.

9. Let P be the vertex of a right circular cone (vertex angle is $45°$), C be the center of the circular base, and K be a point on the outline of the base (i.e. the circle that outlines the base) such that the segment $\overline{CK}$ is parallel to the z axis, as show in the diagram below. This cone has its principle axis (PC) rotated $45°$ from the x axis in the xy plane. What is the angle between the x axis and the segment PK in radians?



10. Positive real numbers x, y, and z satisfy the equations

$$\begin{aligned} x^2 + y^2 &= 9 \\ y^2 + \sqrt{2}yz + z^2 &= 16 \\ z^2 + \sqrt{2}zx + x^2 &= 25. \end{aligned}$$

Compute $\sqrt{2}xy + yz + zx$.

11. Find the volume of the region given by the inequality

$$|x + y + z| + |x + y - z| + |x - y + z| + |-x + y + z| \le 4.$$

12. Suppose we have a polyhedron consisting of triangles and quadrilaterals, and each vertex is shared by exactly 4 triangles and one quadrilateral. How many vertices are there?

13. A one-dimensional ladder of length c is restricted so that one endpoint must lie on the positive x axis and the other endpoint on the positive y axis. Let f be the curve that is tangent to each of the possible arrangements of ladders. Find an equation for f.

14. A, B, C, D are points along a circle, in that order. AC intersects BD at X. If $BC = 6$, $BX = 4$, $XD = 5$, and $AC = 11$, find AB.

15. There are five balls that look identical, but their weights all differ by a little. We have a balance that can compare only two balls at a time. What is the minimum number of times we have to use to balance to rank all balls by weight?