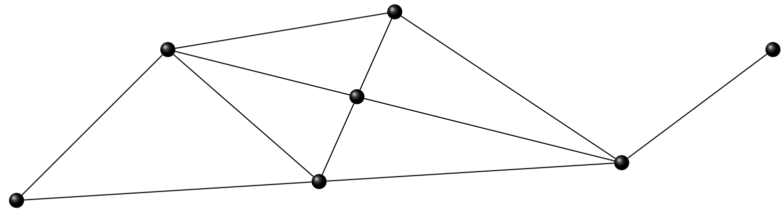
2010 RMT / SMT Power Round

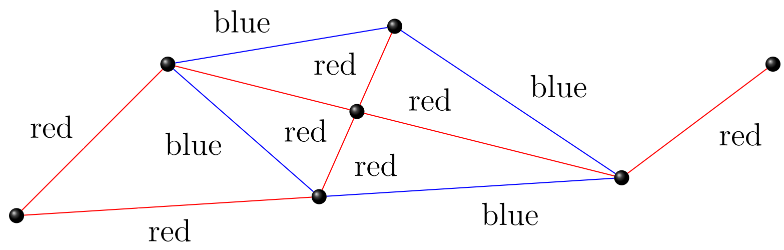
Definitions

A *graph* is a collection of points (vertices) connected by line segments (edges). In this test, all graphs will be *simple* – any two vertices will be connected by at most one edge – and *connected* – you can get from any vertex to any other by following edges.



A simple graph with 7 vertices and 11 edges.

An *edge n-coloring* of a graph G is an assignment of one of n colors to each edge of G.



A 2-coloring of the earlier graph

A *complete* graph is one in which any two vertices are connected by an edge.

1. a. Draw a simple connected graph with 8 vertices and 7 edges, and 3-color its edges.
   b. Draw a complete graph on 5 vertices, and 2-color its edges so that it does *not* contain a red triangle or a blue triangle (3 vertices, the edges between which are all red or all blue).

We will use K_n to denote a complete graph on n vertices. A *monochromatic* K_n is one in which every edge has the same color. Hence, problem 1(b) could have been phrased "Color K_5 so that it has no monochromatic K_3".

2. Show that no matter how you 2-color K_6, it will contain a monochromatic K_3. (Hint: Think about all the edges coming from one vertex).

The *Ramsey number* $R(k)$ is the least number n such that no matter how you 2-color the edges of K_n, there will be a monochromatic K_k. In problems 1(b) and 2, you have shown that $R(3) = 6$.

Interestingly, $R(4)$ is a difficult quantity to calculate, and $R(5)$ is still unknown! Since we cannot go much further in this vein, let us try looking at generalizations of Ramsey numbers. Define $R(k, j)$ as the least n such that every red, blue edge 2-coloring of K_n contains either a red K_k or a blue K_j. Then $R(n)$ is just $R(n, n)$ under this new definition.

3. a. Show that $R(3, 4) > 8$ by exhibiting a 2-coloring.
   b. Show that $R(4, 3) = 9$ (Hint: Use problem 2.)

4. a. Show that

$$R(n,m) \leq R(n,m-1) + R(n-1,m).$$

(Hint: see hint to problem 2.)

b. Conclude that $R(n,m)$ is well defined, that is, that it exists for every n and every m.

From here on, we will explore some interesting properties and generalizations of Ramsey numbers. Each section is independent.

Bounds on Ramsey Numbers

5. Color a graph of n^2 points, laid out in a $n \times n$ grid, as follows: The edge (u,v) is blue if u and v are in the same row, and red otherwise.
   a. Show that any K_{n+1} in that graph contains at least one red edge and at least one blue edge.
   b. Conclude that $R(n+1,n+1) > n^2$.

Problem 6 gives us a polynomial lower bound for $R(n,n)$, and it does so constructively – we know exactly which graph will give a counterexample. Erdős has shown that, if we are willing to be nonconstructive, we can get a much better lower bound:

6. a. Show that if the edges of K_m are colored red or blue randomly with equal probability (i.e., by flipping a coin for each edge), then the probability that it contains a monochromatic K_n is at most

$$\binom{m}{n} \cdot 2^{1-\binom{n}{2}}.$$

   b. Show that if $\binom{m}{n} < 2^{\binom{n}{2}-1}$, that probability is less than 1.
   c. Using the fact that $\binom{m}{n} < m^n$, show that if $m = 2^{\frac{n}{2}-\frac{1}{n}-\frac{1}{2}}$ then $\binom{m}{n} < 2^{1-\binom{n}{2}}$, and conclude that

$$R(n,n) > 2^{\frac{n}{2}-\frac{1}{n}-\frac{1}{2}}.$$

7. Prove a complementary upper bound: $R(n,n) \leq 4^n$.

k-color Ramsey Numbers

Similar to our definition $R(n,m)$, we can define $R(n_1,n_2,n_3,\ldots,n_k)$ to be the least m such that if K_m is colored with k colors, there is some monochromatic K_{n_i} of color c_i.

8. Prove that $R(3,3,3) \leq 17$. (In fact, $R(3,3,3) = 17$, but this is difficult to show.)

9. Show that

$$R(n_1,\ldots,n_k) \leq R(n_1,n_2,\ldots n_{k-2}, R(n_k,n_{k-1}))$$

This gives us the existence of $R(n_1,\ldots,n_k)$ for all $\{n_1,\ldots,n_k\}$.

10. Prove that
    a.

$$R(\underbrace{3,\ldots,3}_{r\text{ 3's}}) \leq 3r!$$

b.

$$R(\underbrace{3,\ldots,3}_{r\text{ 3's}}) > 2^r$$

Infinite Ramsey Numbers

11. Define $K_{\mathbb{N}} = (V, E)$, where $V = \{1, 2, 3, \ldots\}$, and $E = \{(i, j) : i, j \in V,\ i < j\}$. This is in some sense an infinite complete graph. Show that if every edge is colored red or blue, there is some infinite subset V' of V such that all of the edges between points of V' are the same color.