1. Answer: 2520

\[ \frac{8!}{(2!)^4} = \frac{7!}{2} = 2520 \]

2. Answer: \(3(a - b)(b - c)(c - a)\)

The expression is zero when any two of \(a, b,\) and \(c\) are equal. So it must have \((a - b)(b - c)(c - a)\) as a factor. But the original polynomial is degree 3, and so is this one, so the remaining factor must be a constant. The original polynomial contains a term \(3ab^2\), but \((a - b)(b - c)(c - a)\) only contains a term \(ab^2\), so the constant must be 3.

3. Answer: 13

Consider the equation modulo 5. All fourth powers are either 0 or 1 mod 5. So one of \(x\) and \(y\) must be divisible by 5; suppose it’s \(x\). Then we must in fact have \(x = 5\), since \(x = 10\) is too large. This gives \(y = 8\), and this is the only possible solution. So the answer is 13.

4. Answer: 41

A recursion relationship describing this problem is

\[ a_1 = 1, a_2 = 2, a_{2n} = a_n + a_{n-1}, a_{2n+1} = a_n \]

where \(a_n\) is the number of valid sums for \(n\). Thus,

\[ a_{657} = a_{328} = a_{163} = a_{82} + 2a_{31} = a_{41} + 3a_{40} = 3a_{19} + 4a_{20} = 4a_{10} + 7a_9 = 4a_5 + 11a_4 = 4a_2 + 11a_4 = 4 \cdot 2 + 11 \cdot 3 = 41. \]

5. Answer: 120

Define \(\Gamma(n) = \int_0^{\infty} t^{n-1}e^{-t}dt\). Using integration by parts,

\[ \Gamma(n + 1) = \int_0^{\infty} t^n e^{-t}dt \]

\[ = -t^n e^{-t}\bigg|_0^\infty + \int_0^{\infty} nt^{n-1}e^{-t}dt \]

\[ = 0 + n \int_0^{\infty} t^{n-1}e^{-t}dt \]

\[ = n \Gamma(n). \]

Next we evaluate \(\Gamma(1) = \int_0^{\infty} e^{-t}dt = -e^{-t}\bigg|_0^\infty = 0 - 1 = 1\). Thus, \(\Gamma(n + 1) = n \Gamma(n) = \ldots = n! \Gamma(1) = n!\). So for the problem, \(\Gamma(6) = 5! = 120\).

6. Answer: \(\frac{2}{\pi}\)

Area of Rhombus \(ABCD\) = \[4 \times \frac{1}{2} \times \cos \frac{\theta}{2} \sin \frac{\theta}{2}\]

\[= 2 \times \cos \frac{\theta}{2} \sin \frac{\theta}{2} = \sin \theta\]

\[E[\text{Rhombus } ABCD] = \frac{1}{\pi} \int_0^{\pi} 2 \sin \theta dx\]

\[= \frac{2}{\pi} \times 1\]

\[= \frac{2}{\pi}.\]
7. **Answer:** $\frac{3\sqrt{3}}{4}$

Let the angle between the longer base and the leg be $\theta$.

The Area of the Trapezoid $\Delta(\theta) = \sin \theta + \sin \theta \cdot \cos \theta = \sin \theta + \frac{1}{2} \sin 2\theta$

The area reaches extrema when its derivative is zero:

$\Delta' = \cos \theta + \cos 2\theta = 0$

We use the formula $\cos 2\theta = 2 \cdot \cos^2 \theta - 1$

$\cos \theta = \frac{-1 \pm \sqrt{9}}{4} = \frac{1}{2} \text{ or } -1 \text{ (omitted)}$

$\sin \theta = \frac{\sqrt{3}}{2}$

$\Delta_{\text{Max}} = \left[ \frac{3\sqrt{3}}{4} \right]$

8. **Answer:** $\frac{-n^2 + 1}{12n^2}$

$$\sum_{k=1}^{n} \frac{k^2(k - n)}{n^4} = \sum_{k=1}^{n} \frac{k^3 - k^2n}{n^4} = \sum_{k=1}^{n} \frac{k^3}{n^4} - \sum_{k=1}^{n} \frac{k^2}{n^3} = \frac{1}{n^4} \sum_{k=1}^{n} k^3 - \frac{1}{n^3} \sum_{k=1}^{n} k^2 = \left( \frac{1}{n^4} \right) \left( \frac{n(n + 1)}{2} \right)^2 - \left( \frac{1}{n^3} \right) \left( \frac{n(n + 1)(2n + 1)}{6} \right) = \frac{n^4 + 2n^3 + n^2}{4n^4} - \frac{2n^3 + 3n^2 + n}{6n^3} = \frac{-n^4 + n^2}{12n^4} = \frac{-n^2 + 1}{12n^2}$$

9. **Answer:** $2\sqrt{17}$

Find the point on the parabola closest to the point (6,12). Call it $(x, y)$ This point is where the normal line at $x$ crosses the parabola. We find the derivative by:

$$x = y^2$$

$$\frac{dx}{dy} = y$$

$$\frac{dy}{dx} = \frac{1}{y}$$

The normal line will have slope of $-y$. It will contain (6,12). Its equation is:

$$y - 12 = -y(x - 6)$$

$$y = -xy + 6y + 12$$

$$y = -\frac{y^3}{2} + 6y + 12$$

$$2y = -y^3 + 12y + 24$$

$$0 = y^3 - 10y - 24$$
The roots are 4 and two other imaginary answers, so 4 is the only one that works.

\[ y - 12 = -y(x - 6) \]
\[ -8 = -4(x - 6) \]
\[ x = 8 \]

Find the distance between (8, 4) and (6, 12). The answer is \(2\sqrt{17}\).

10. **Answer: 2**

More generally, define a function \(G\) by

\[ G(m) = \sum_{n=m}^{\infty} \frac{\binom{n}{m}}{2^n}. \]

Thus we wish to evaluate \(G(2009)\). Observe that for all \(m \geq 1:\)

\[
G(m) = \sum_{n=m}^{\infty} \frac{\binom{n}{m}}{2^n} = \sum_{n=m}^{\infty} \frac{\binom{n-1}{m-1} + \binom{n-1}{m}}{2^n} = \frac{1}{2} \sum_{n=m}^{\infty} \frac{\binom{n}{m-1}}{2^n} + \frac{1}{2} \sum_{n=m-1}^{\infty} \frac{\binom{n}{m}}{2^n} = \frac{1}{2} (G(m-1) + G(m))
\]

And thus \(G(m) = G(m-1)\). Thus it suffices to evaluate \(G(0)\). However, this is simply a geometric series:

\[ G(0) = \sum_{n=0}^{\infty} \frac{1}{2^n} = 2. \]

**NOTE:** By noticing that \(\binom{n}{n_{2009}}\) is \(\frac{1}{n!}n^{2009}\) asymptotically, one can see this summation as a discrete analogue of the Euler \(\Gamma\) function, which is defined by \(\Gamma(x) = \int_0^{\infty} \frac{t^{x-1}}{e^t} dt\). The solution above is similar to the proof that \(\Gamma(n + 1) = n\Gamma(n)\).

11. **Answer: 1266**

\[
(1 + z_1^2 z_2)(1 + z_1 z_2^2) = 1 + z_1^2 z_2 + z_1 z_2^2 + z_1^3 z_2^3 = 1 + z_1 z_2 (z_1 + z_2) + (z_1 z_2)^3.
\]

Since \(z_1 + z_2 = -6\) and \(z_1 z_2 = 11\),

\[
(1 + z_1^2 z_2)(1 + z_1 z_2^2) = 1 + 11(-6) + 11^3 = 1266.
\]
12. Answer: 13689

\[ 2009 = 7^2 \times 41 \]

We know for a number \( n = a_1^{\alpha_1} \times a_2^{\alpha_2} \times \ldots \times a_n^{\alpha_n} \), it has \((\alpha_1 + 1) \times (\alpha_2 + 1) \times \ldots (\alpha_n + 1) \) factors.

Hence, for number N, we have the following options:

\[ \alpha_1 = 7 - 1 = 6, \quad \alpha_2 = 7 \times 41 - 1 = 289 - 1 = 288 \]
\[ \alpha_3 = 41 - 1 - 40 \]

By the same fact mentioned above, \( N^2 \) has: \((2 \times \alpha_1 + 1) \times (2 \times \alpha_2 + 1) \times \ldots (2 \times \alpha_n + 1) \) factors.

Calculating this number for both, we get the 2nd option gets us a bigger number: \( 13 \times 13 \times 81 = 13689 \)

13. Answer: 3

\[ 17^{289} \equiv (14 + 3)^{289} \equiv \binom{289}{1} 14^{288} 3 + \ldots + \binom{289}{n} 13^{289-n} 3^n + \ldots \]
\[ 3^{289} \equiv 3^{289} \pmod{7} \]

Note that \( 3^3 \equiv 27 \equiv -1 \pmod{7} \). Then \( 3^{289} \equiv 3^{3^9} 3^1 \equiv (-1)^9 3^1 \equiv 3 \pmod{7} \). Thus, the remainder is 3.

14. Answer: 17

Equation modulo 23, we get \( -6(a - b) \equiv -10 \pmod{23} \). Since -4 is an inverse of -6 modulo 23, then we multiply to get \((a - b) \equiv 17 \pmod{23} \). Therefore, the smallest possible positive value for \((a - b)\) is 17. This can be satisfied by \( a = 5, b = -12 \).

15. Answer: 66

\[ \left\lfloor \frac{2008}{31} \right\rfloor + \left\lfloor \frac{2008}{31^2} \right\rfloor + \left\lfloor \frac{2008}{31^3} \right\rfloor + \left\lfloor \frac{2008}{31^4} \right\rfloor + \ldots = 64 + 2 + 0 + 0 + \ldots = 66. \]