1. Answer: $\frac{1}{324}$
   The conditions imply that $|a - 1| = |b - 2| = |c - 3| = |d - 6| = 1$. $a$ can equal 2, $b$ can equal 1 or 3, $c$ can equal 2 or 4, and $d$ can equal 5. So the probability is $\frac{1}{5} \times \frac{2}{5} + \frac{2}{5} \times \frac{1}{5} = \frac{1}{324}$.

2. Answer: 3, 7
   Zero is the only digit with square ending in 0. The square of a number ending in zero will therefore end in two zeros. Next digit of the number therefore needs a square ending in 9, so it is 3 or 7.

3. Answer: $\frac{147}{52}$
   The number of teeth meshed does not vary. Thus, if $n$ is the number of revolutions that gear 10 make, then $(5(1) + 21)(21) = (5(10) + 2)n \Rightarrow n = \frac{7 \times 21}{52} = \frac{147}{52}$.

4. Answer: 245
   If we make our team all the same type, then there are $\binom{8}{2} + \binom{6}{2} + \binom{6}{2} = 7 + 1 + 28 = 36$ ways to do this. If we make our team partially bug and partially rock type, there are $\binom{6}{2} \binom{4}{1} + \binom{5}{2} \binom{4}{1} = 15 \times 1 + 20 \times 4 + 45 \times 6 + 4 \times 4 = 15 + 80 + 90 + 24 = 209$ ways. Any other combination of types will not work. This gives a total of 245 ways.

5. Answer: 1820, or $\binom{13}{1} + 3\binom{13}{2} + 3\binom{13}{3} + \binom{13}{4}$
   We proceed by casework.
   Case 1 All cards have the same face value. There are $\binom{13}{1}$ ways to choose the face values.
   Case 2 Some cards have face value $A$: some have face value $B$. There are $\binom{13}{2}$ ways to choose $A$ and $B$. One can have the combinations $ABBB$, $AABB$, $AAAB$, so there are $3\binom{13}{2}$ distinct ways for this case.
   Case 3 Some cards have value $A$, some $B$, and some $C$. There are $\binom{13}{3}$ ways to choose the $A$, $B$, $C$. One can have the combinations $ABCC$, $ABBC$, and $AABC$. There are $3\binom{13}{3}$ distinct ways for this case.
   Case 4 The cards are distinct: $ABCD$. There are $\binom{13}{4}$ ways to do this. Since these cases are mutually exclusive, we have $\binom{13}{1} + 3\binom{13}{2} + 3\binom{13}{3} + \binom{13}{4} = 1820$ distinct hands.

6. Answer: $0, \pm 2, 1 \pm i\sqrt{3}, -1 \pm i\sqrt{3}$
   Clearly 0 is a solution. Now we assume $z \neq 0$. We have $|z^5| = |16z|$. By DeMoivre's Theorem, $|z|^5 = |z|^5$. The left hand side becomes $|z|^5 = 16|z| = 16$ $|z|$. Equating the two sides, $16|z| = |z|^5 \Rightarrow |z|^4 = 16 \Rightarrow |z| = 2$.
   Multiplying both sides of the given equation by $z$,
   $$z^6 = 16|z|^2 = 64.$$  
   Let $z = r(\cos \theta + i\sin \theta)$. Then $r^6(\cos(6\theta) + i\sin(6\theta)) = 64$. Thus, $r = 2$ and $6\theta = 360k$, for $k = 0, 1, 2, 3, 4, 5$. So our other solutions are $2, 2\cos(60^\circ), 2\cos(120^\circ), -2, 2\cos(240^\circ), 2\cos(300^\circ)$, which are equal to $\pm 2, 1 \pm i\sqrt{3}, -1 \pm i\sqrt{3}$.

7. Answer: $e^{\pi i}$ or $\pm \frac{\sqrt{3} + i}{2}$
   Let $x = \sqrt{\frac{1 + \sqrt{3}i}{2}}$. Then $x^2 = \frac{1 + \sqrt{3}i}{2}$. Converting to polar form, $\frac{1 + \sqrt{3}i}{2} = (e^{\pi i})^\frac{1}{2} = e^{\frac{\pi i}{2}} = \frac{\sqrt{3} + i}{2}$

8. Answer: $\frac{13}{35}$
   If the first toss comes up heads (2/3 probability), Frank has a 1/4 chance of getting another heads, a (3/4) * (1/3) = 1/4 chance of getting two successive tails, and a (3/4) * (2/3) = 1/2 chance of getting tails-heads and winding up back at his current position of tossing the “1/4–3/4” coin with the previous toss being a heads. Expressing the probabilities as geometric series (or just the weighted probability of the two nonrepeating options), he has a 1/2 chance of getting HH first and a 1/2 chance of getting
TT first. If instead, the first toss comes up tails (1/3 probability), he has a 3/4 chance of getting another tails, a (1/4) * (2/3) = 2/12 chance of getting two successive heads, and a (1/4) * (1/3) = 1/12 chance of getting heads-tails and winding up back at my current state. Expressing the probabilities as a geometric series, he has a 2/11 chance of getting HH first and a 9/11 chance of getting TT first. The probability of getting HH before TT is (2/3) * (1/2) + (1/3) * (2/11) = 13/33.

9. **Answer: 590**

Expanding out the recurrence relations, we confirm that the triangular numbers are $T_n = 1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}$ and the square numbers are $S_n = n^2$. A general formula for the pentagonal numbers is therefore $P_n = n^2 + n(n - 1)/2 = n(3n - 1)/2$. Substituting $n = 20$ gives $P_{20} = 20(60 - 1)/2 = 590$.

10. **Answer: 6**

$e^{i\pi/3} + e^{2i\pi/3} + e^{3i\pi/3} + e^{4i\pi/3} + e^{5i\pi/3} + e^{6i\pi/3}$ sum to 0 because the terms are sixth roots of unity (i.e. they satisfy $z^6 - 1 = 0$, which is a 6th degree polynomial whose 5th degree coefficient is 0). Likewise, $e^{2i\pi/3} + e^{4i\pi/3} + e^{6i\pi/3}$ sum to zero because the terms are cubic roots of unity. $e^{3i\pi/3} + e^{6i\pi/3}$ sum to 0 because they are square roots of unity. Subtracting these sums from the original expression, we are left with only $6e^{6i\pi/3}$, which is $6(\cos(2\pi) + i\sin(2\pi)) = 6$. 