SMT/RMT Power Question 2009

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In this round, we will explore various fun things you can do with dice. Throughout, we will refer to n-sided dice for various values of n. Don't worry about how you could actually construct such dice physically; just think of them as random number selectors with n possible outcomes.

Prove all of your answers. Little or no credit will be given for numerical answers without proof, even if correct.

Part I: Stacking Dice

"Standard dice" in this section refers to six-sided dice, numbered 1-6, with 1 opposite 6, 2 opposite 5, and 3 opposite 4. For all of these problems, ignore the fact that there's a table or other surface under the dice; the numbers on the faces on the bottom of a stack count as being "visible". Dice all have the same chirality, meaning that a die and its mirror image do not both exist, all dice are similar by rotation.

1. Suppose we stack up 7 identical standard dice in a 1*1*7 tower, in such a way that any two faces that touch each other have the same number. Find the sum of the numbers on the visible faces (remember, this includes the bottom of the tower).

2. Now we arrange 4 identical standard dice in a 1*2*2 block, and impose the additional condition that on each of the block's 2*2 faces, the four numbers showing are the same. Find the sum of the numbers on all the visible faces.

3. Now we arrange 8 dice in a 2*2*2 cube. Find all possible values of the sum of the numbers on all the visible faces.

Part II: Non-Transitive Dice

Suppose we have two dice A and B, and if we roll them both, A shows a higher number than B more than half the time. Then we say A is "better" than B.

4. The following six-sided dice are known as Efron's Dice:

   A: 0, 0, x, x, x, x

B: 3, 3, 3, 3, 3, 3

C: 2, 2, 2, 2, 6, 6

D: y, y, y, 1, 1, 1

They have the property that A is better than B, B is better than C, C is better than D, and D is better than A. For this reason, we call them *nontransitive* dice.

a. If x and y are integers, what must they be?

b. Suppose we roll all four dice at once, and the highest number showing wins. Calculate the probability of winning for each die.

5. a. Design a set of three 3-sided non-transitive dice (i.e.A is better than B, B is better than C, and C is better than A). Use the numbers 1 through 9 each once.

   b. Suppose we want to do this in such a way that when all three dice are rolled at once, each has an equal chance of winning. Prove that this is impossible.

   c. Do part (a), but use only the numbers 1-5, and don't use any number on two different dice (i.e. make ties impossible).

6. a. Prove that it is always possible to create a set of $2n+1$ dice (with any number of sides) such that each die is better than n of the others and worse than n of the others, for any positive integer n.

   b. Do this with $n = 2$ (i.e. five dice), and make the dice 3-sided. Use the numbers 1 through 15 each once.

   c. Prove that you can always do this with 3-sided dice, and show how to do it using the numbers 1 through $6n + 3$.

7. Suppose we want to construct four six-sided dice so that A is better than B, B is better than C, C is better than D, and D is better than A (like Efron's Dice). We also want to set them up so that in each of these cases, the winning die wins with at least probability P. Prove that P cannot be more than $\frac{2}{3}$.

Part III: Multiple Rolls

In this section, we examine a game where two players, Alice and Bob, have different dice. They each roll their own die n times and add up the results. Whoever has the highest total wins.

8. Suppose Alice's die rolls 0 half the time and 2 half the time. Bob's die always rolls x, where x is an irrational number between 0 and 1. For what values of n and x does Bob have a better chance of winning than Alice? For what values does Alice have a better chance? For what values do they each have an equal chance?

9. Now suppose that $\frac{2}{3}$ of the time Alice's die rolls 1, and the other $\frac{1}{3}$ of the time it rolls -2, while Bob always rolls 0. Assume $n = 3m$ for an integer m.

   Prove that at the end, Alice's score is always divisible by 3.