1. **Answer:** $\mathbf{\frac{1}{324}}$

   The conditions implies that $|a-1| = |b-2| = |c-3| = |d-6| = 1$. a can equal 2, b can equal 1 or 3, c can equal 2 or 4, and d can equal 5. So the probability is $\frac{1}{6} * \frac{2}{6} * \frac{2}{6} * \frac{1}{6} = \frac{1}{324}$.

2. **Answer: 3, 7**

   Zero is the only digit with square ending in 0. The square of a number ending in zero will therefore end in two zeros. Next digit of the number therefore needs a square ending in 9, so it is 3 or 7.

3. **Answer:** $\mathbf{\frac{147}{52}}$

   The number of teeth meshed does not vary. Thus, if n is the number of revolutions that gear 10 make, then $(5(1)+21)(21) = (5(10)+2)n \Rightarrow n = \frac{7\times 21}{52} = \frac{147}{52}$.

4. **Answer: 245**

   If we make our team all the same type, then there are $\binom{7}{6} + \binom{6}{6} + \binom{4}{6} + \binom{8}{6} = 7+1+0+28 = 36$ ways to do this. If we make our team partially bug and partially rock type, there are $\binom{6}{2}\binom{4}{4} + \binom{6}{3}\binom{4}{3} + \binom{6}{4}\binom{4}{2} + \binom{6}{5}\binom{4}{1} = 15*1+20*4+15*6+6*4 = 15+80+90+24 = 209$ ways. Any other combination of types will not work. This gives a total of 245 ways.

5. **Answer: 1820, or** $\mathbf{\binom{13}{1} + 3\binom{13}{2} + 3\binom{13}{3} + \binom{13}{4}}$

   We proceed by casework.
   Case 1 All cards have the same face value. There are $\binom{13}{1}$ ways to choose the face values.
   Case 2 Some cards have face value A; some have face value B. There are $\binom{13}{2}$ ways to choose A and B. One can have the combinations $ABBB$, $AABB$, $AAAB$, so there are $3\binom{13}{2}$ distinct ways for this case.
   Case 3 Some cards have value A, some B, and some C. There are $\binom{13}{3}$ ways to choose the A, B, C. One can have the combinations $ABCC$, $ABBC$, and $AABC$. There are $3\binom{13}{2}$ distinct ways for this case.
   Case 4 The cards are distinct: $ABCD$. There are $\binom{13}{4}$ ways to do this. Since these cases are mutually exclusive, we have $\binom{13}{1} + 3\binom{13}{2} + 3\binom{13}{3} + \binom{13}{4} = 1820$ distinct hands.

6. **Answer:** $\mathbf{0, \pm 2, 1 \pm i\sqrt{3}, -1 \pm i\sqrt{3}}$

   Clearly 0 is a solution. Now we assume $z \neq 0$. We have $|z^5| = |16\bar{z}|$. By DeMoivre's Theorem, $|z^5| = |z|^5$. The left hand side becomes $|z^5| = 16|\bar{z}| = 16|z|$. Equating the two sides, $16|z| = |z|^5 \Rightarrow |z|^4 = 16 \Rightarrow |z| = 2$.
   Multiplying both sides of the given equation by z,

   $$z^6 = 16|z|^2 = 64.$$

   Let $z = r(\cos\theta + i\sin\theta)$. Then $r^6(\cos(6\theta) + i\sin(6\theta)) = 64$. Thus, $r = 2$ and $6\theta = 360k$, for $k = 0, 1, 2, 3, 4, 5$. So our other solutions are $2, 2\text{cis}(60°), 2\text{cis}(120°), -2, 2\text{cis}(240°), 2\text{cis}(300°)$, which are equal to $\pm 2, 1 \pm i\sqrt{3}, -1 \pm i\sqrt{3}$.

7. **Answer:** $\mathbf{e^{\frac{\pi}{6}i}}$**, or** $\mathbf{\pm\frac{\sqrt{3}+i}{2}}$

   Let $x = \sqrt{\frac{1+\sqrt{3}i}{2}}$. Then $x^2 = \frac{1+\sqrt{3}i}{2}$. Converting to polar form, $\frac{1+\sqrt{3}i}{2} = (e^{\frac{\pi}{3}i})^{\frac{1}{2}} = e^{\frac{\pi}{6}i} = \frac{\sqrt{3}+i}{2}$

8. **Answer:** $\mathbf{\frac{13}{33}}$

   If the first toss comes up heads (2/3 probability), Frank has a 1/4 chance of getting another heads, a $(3/4)*(1/3) = 1/4$ chance of getting two successive tails, and a $(3/4)*(2/3) = 1/2$ chance of getting tails-heads and winding up back at his current position of tossing the "1/4−3/4" coin with the previous toss being a heads. Expressing the probabilities as geometric series (or just the weighted probability of the two nonrepeating options), he has a 1/2 chance of getting HH first and a 1/2 chance of getting

TT first. If instead, the first toss comes up tails (1/3 probability), he has a 3/4 chance of getting another tails, a $(1/4) * (2/3) = 2/12$ chance of getting two successive heads, and a $(1/4) * (1/3) = 1/12$ chance of getting heads-tails and winding up back at my current state. Expressing the probabilities as a geometric series, he has a 2/11 chance of getting HH first and a 9/11 chance of getting TT first. The probability of getting HH before TT is $(2/3) * (1/2) + (1/3) * (2/11) = 13/33$.

9. **Answer: 590**

   Expanding out the recurrence relations, we confirm that the triangular numbers are $T_n = 1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}$ and the square numbers are $S_n = n^2$. A general formula for the pentagonal numbers is therefore $P_n = n^2 + n(n-1)/2 = n(3n-1)/2$. Substituting $n = 20$ gives $P_{20} = 20(60-1)/2 = 590$.

10. **Answer: 6**

    $e^{i\pi/3} + e^{2i\pi/3} + e^{3i\pi/3} + e^{4i\pi/3} + e^{5i\pi/3} + e^{6i\pi/3}$ sum to 0 because the terms are sixth roots of unity (i.e. they satisfy $z^6 - 1 = 0$, which is a 6th degree polynomial whose 5th degree coefficient is 0). Likewise, $e^{2i\pi/3} + e^{4i\pi/3} + e^{6i\pi/3}$ sum to zero because the terms are cubic roots of unity. $e^{3i\pi/3} + e^{6i\pi/3}$ sum to 0 because they are square roots of unity. Subtracting these sums from the original expression, we are left with only $6e^{6i\pi/3}$, which is $6(\cos(2\pi) + i\sin(2\pi)) = 6$.