1. **Answer:** $\boldsymbol{e}$

   The exponent simplifies to $\frac{1}{2}\sin(2x)$, which has maximum value $\frac{1}{2}$.

2. **Answer:** $\boldsymbol{z = 7}$

   Let $\frac{d}{dz}(56z - 4z^2) = 56 - 8z = 0 \Rightarrow 8z = 56 \Rightarrow z = 7$

3. **Answer: 4**

   The first few terms of the sequence are $10, 5, 12, 6, 3, 8, 4, 2, 1, 4, 2, 1, \ldots$. Since the sequence repeats every three terms and 2008 is one more than a multiple of 3, the 2008^{th} term has the same value as the 7^{th} term.

4. **Answer:** $\boldsymbol{2\pi}$

   We integrate using the disk method:

$$V = \pi \int_a^b R(x)^2 dx = \pi \int_0^2 \left(\sqrt{x}\right)^2 dx = \pi \int_0^2 x dx = 2\pi$$

5. **Answer:** $\boldsymbol{\frac{1}{6}}$

   Remember that a regular octahedron is composed of two square pyramids. If the vertices are at the centers of the faces of a unit cube, then the vertices of the bases of these pyramids are at the centers of the sides of a unit square. Such a base has area $\frac{1}{2}$. The height of each pyramid is also $\frac{1}{2}$, which gives a volume for each pyramid of $\frac{1}{3} \cdot \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{12}$. The total volume is thus $\frac{1}{6}$.

6. **Answer:** $\boldsymbol{\frac{1}{2}\sqrt{x^2y^2 + x^2z^2 + y^2z^2}}$

   Let $\vec{u} = (-x, y, 0)$ and $\vec{v} = (-x, 0, z)$ be the vectors from the first point to the other two. The area of the triangle is then half the magnitude of the cross product:

$$\frac{1}{2}\left|\vec{u} \times \vec{v}\right| = \frac{1}{2}\left|(yz, xz, xy)\right|$$

7. **Answer:** $\boldsymbol{\frac{7}{15}}$

   Let the interior intersections from A, B, C, D be a, b, c, d, respectively. Let $P(x_Y)$ be the probability of exiting at Y when currently at x. Then, because we enter at A, we are trying to find $P(a_A) = \frac{1}{3}(1) + \frac{1}{3}P(b_A) + \frac{1}{3}P(d_A)$. Similarly, $P(b_A) = \frac{1}{3}(0) + \frac{1}{3}P(a_A) + \frac{1}{3}P(c_A)$ and $P(d_A) = \frac{1}{3}(0) + \frac{1}{3}P(a_A) + \frac{1}{3}P(c_A)$ and $P(c_A) = \frac{1}{3}(0) + \frac{1}{3}P(b_A) + \frac{1}{3}P(d_A)$. Noting the symmetry between b and d, we have

$$\begin{cases} P(a_A) = \frac{1}{3}(1) + \frac{2}{3}P(b_A) \\ P(b_A) = \frac{1}{3}(0) + \frac{1}{3}P(a_A) + \frac{1}{3}P(c_A) \\ P(c_A) = \frac{1}{3}(0) + \frac{2}{3}P(b_A) \end{cases}$$

   This is a system of three equations with three unknowns. Solving for $P(a_A)$ yields the required probability of 7/15.

8. **Answer:** $\boldsymbol{-\frac{\sqrt{13}}{13}}$

   Let $AE = EP = AP = 1$. Then $AP = PT = 1$, so $\triangle APT$ is isosceles. Since $m\angle APT = 120°$, $m\angle PAT = 30°$, so $\angle TAE$ is a right angle! This makes $\triangle ETA$ a 30-60-90 triangle, so $AT = \sqrt{3}$. Since $TX = 2$ and $m\angle PTA = 30°$, $m\angle XTA = 150°$, and we can use the Law of Cosines to find AX.

   The Law of Cosines gives $AX^2 = 3 + 4 - 4\sqrt{3}\cos(150°) = 7 + 4\sqrt{3} \cdot \frac{\sqrt{3}}{2} = 13$.

   Now use the Law of Cosines again to get $\cos(m\angle XAE) = \frac{AX^2 + AE^2 - EX^2}{2 \cdot AX \cdot AE} = \frac{-2}{2\sqrt{13}} = -\frac{\sqrt{13}}{13}$.

9. **Answer: $\frac{1}{n+1}$**

   Put $XKCD$ in the coordinate plane with X at the origin, K on the positive y-axis, and D on the positive x-axis. Let $XK = t$, and scale the rectangle so that $KJ = 1$ (and thus $KC = n$). Then the length of $\overline{QZ}$ is just the x-coordinate of point Q, since it's perpendicular to the y-axis. Line $\overline{DK}$ is given by the equation $y = -\frac{t}{n}x + t$, and $\overline{XJ}$ is given by $y = tx$. Q is their intersection point, so the x-coordinate of Q is the solution to $-\frac{t}{n}x + t = tx$. Multiply through by $\frac{n}{t}$ to get $-x + n = nx$, which gives $x = \frac{n}{n+1}$. Since $XD = KC = n$, $\frac{XD}{QZ} = n + 1$.

10. **Answer: $\frac{16}{\binom{8}{4}}$ or $\frac{8}{35}$**

    Notice that if we just consider cards A and B, there are four classes of numbers: 2 and 7 each appear on both A and B; 3 and 5 appear only on A; 4 and 6 appear only on B; and 1 and 8 appear on neither A nor B. So, using information from cards A and B alone, Bill can identify which of these four classes the volunteer's number belongs to, but nothing more. Thus, in order for Bill to be able to perform his trick, he needs to be able to use card C to split each class in half; hence, C must contain one number from each of the four classes. Thus, there are 2^4 choices of numbers that Bill can write that will allow him to perform his trick. Since there are $\binom{8}{4}$ choices available to Bill, the probability that he will be able to perform his trick is $\frac{2^4}{\binom{8}{4}} = \frac{8}{35}$.

11. **Answer: $(49, 4)$**

    Note: if you found a creative method to find a root besides $(16, -7)$ and $(49, 4)$ and brought it to our attention as a protest, we would have accepted it. On contest day, we did not exclude $(16, -7)$, making the question extremely easy, so it was thrown out.

    Let $g(x, y) = f(x, \sqrt{y}) = x^7y^4 + x^4y^7 + A$. g is then symmetric in x, y, so $0 = f(16, 7) = g(16, 49) = g(49, 16) = f(49, 4)$, and therefore $(49, 4)$ is a root.

12. **Answer: 24**

    Label the vertices of the octagon $ABCDEFGH$. Then $ABCD$ is a trapezoid, and there are 7 others congruent to it. $ACDF$ is also a trapezoid, and there are 7 others congruent to it. Finally, $ABDE$ is a trapezoid, and there are 7 congruent to it, making a total of 24 trapezoids.

13. **Answer: $20°, 60°, 100°, 140°, 220°, 260°, 300°, 340°$ or $\frac{\pi}{9}, \frac{\pi}{3}, \frac{5\pi}{9}, \frac{7\pi}{9}, \frac{11\pi}{9}, \frac{13\pi}{9}, \frac{5\pi}{3}, \frac{17\pi}{9}$**

    The left side is equal to $\frac{x^9+1}{x+1}$, so we must have $x^9 = -1$ and $x \neq -1$. This gives $9\theta = 360n + 180$ for some integer n, which gives $\theta = 20$, 60, 100, 140, 220, 260, 300, or 340.

14. **Answer: $n < 1$**

    Let $u = \tan(x)$, so $du = \sec^2(x)dx = (1 + u^2)du$. Then the integral becomes $\displaystyle\int_{-\infty}^{\infty} \frac{u^n}{1 + u^2}\, dx$. For large u, this function behaves almost exactly like u^{n-2}, so if $n < 1$ it's integrable, and if $n > 1$, it's not. If $n = 1$, the function's antiderivative is $\frac{1}{2}\ln(1 + u^2)$, but the limits of this function are ∞ as u goes to $\pm\infty$, so the integral is undefined. Thus we must have $n < 1$.

15. **Answer: 12**

    At $t = 0$, $(x, y) = (1, 0)$. This parametric equation makes a spiral where the loops do not touch. Hence, every 2π steps, there will be an intersection, which corresponds to π steps in the x-axis. Since the spiral intersects itself at a small distance before each $1 + 2\pi n$, there are $39/\pi \approx 12$ intersections.