Geometry Solutions
2007 Rice Math Tournament
February 24, 2007

1. **Answer: $4\sqrt{3}$**

   Let the side length be s. $3s = \frac{\sqrt{3}s^2}{4} \Rightarrow 12s = \sqrt{3}s^2 \Rightarrow s = \frac{12}{\sqrt{3}} = 4\sqrt{3}$

2. **Answer: $8\sqrt{3}$**

   Since each side of the octahedron is a radius of a sphere, the surface area is the area of 8 equilateral triangles with side length 2. $\frac{2\cdot\sqrt{3}}{2} \cdot 8 = 8\sqrt{3}$.

3. **Answer: $\frac{s}{2}$**

   Let A, B be the apexes of two pyramids over adjacent faces, and let O lie at the center of the cube. For the cumulation to form rhombuses instead of tetrahedrons, the hypotenuse of the right triangle OAB must intersect the shared edge of the two faces at a point P. OP is then an altitude, with length $s\sqrt{2}/2$, and so OA is s, with $s/2$ lying inside the cube, leaving $h = s/2$.

4. **Answer: 12**

   Let M be the endpoint of the altitude on the hypotenuse. Since we are dealing with right triangles, $\Delta MAC \sim \Delta ABC$, so $AM = 12/5$. Let N be the endpoint he reaches on side $\overline{AC}$. $\Delta MAC \sim \Delta NAM$, so $\frac{MN}{AM} = 4/5$. This means that each altitude that he walks gets shorter by a factor of $4/5$. The total distance is thus $\frac{12}{5} / \left(1 - \frac{4}{5}\right) = 12$.

5. **Answer: $\frac{s\sqrt{6}}{6}$**

   Let ABC be the vertices on one face of the octahedron, and O be the center of the octahedron (and the circle). Let M be the midpoint of $\overline{AB}$ and P be the center of ΔABC. Then $OM = \frac{s}{2}$, and $MP = \frac{1}{3} \cdot \frac{s\sqrt{3}}{2}$, and $\overline{OP} \perp \overline{MP}$. So $\overline{OP} = s\sqrt{(\frac{l}{2})^2 - (\frac{s\sqrt{3}}{2})^2} = \frac{s}{\sqrt{6}}$.

6. **Answer: $\frac{12\sqrt{5}}{5}$**

   Extend $\overline{IR}$ and $\overline{CE}$ to point X. Clearly, $m\angle X = 90$. Note that $\Delta XIE \sim \Delta XCI$. Since $(XC)/(XI) = 4/8 = 1/2$, we can use the Pythagorean Theorem to solve for XC and XI to get $XC = \frac{4}{\sqrt{5}}$ and $XI = \frac{8}{\sqrt{5}}$. Since $XC/XI = XI/XE$, $XE = 16\sqrt{5}/5$, and $CE = XE - XC = \frac{12\sqrt{5}}{5}$.

7. **Answer: $\frac{\sqrt{2}}{3}$**

   The regions in which they do NOT overlap are the smaller tetrahedra of side length 1 positioned at 3 vertices of both larger tetrahedra. The volume of the overlap region is the volume of one tetrahedron of side length 2, minus the volume of four tetrahedra of side length 1.

   In general, the volume of a tetrahedron of side length $\frac{s}{\sqrt{2}}$ can be found by noting that a cube consists of regular tetrahedron and 4 triangular pyramids (formed from a vertex of the cube and 3 adjacent vertices). The cube has side length s, so the volume of the tetrahedron is $s^3 - 4 \cdot \frac{\frac{s^2}{2} \cdot s}{3} = \frac{1}{3}s^3$, or $1/3$ the volume of a cube of side length s. So the volume of a tetrahedron of side length 1 is $\frac{1}{3} \cdot \frac{1}{\sqrt{8}} \cdot (1)^3 = \frac{\sqrt{2}}{12}$. So the total volume of the region we are looking for is $\frac{\sqrt{2}}{12} \cdot 2^3 - \frac{4\sqrt{2}}{12} = \frac{\sqrt{2}}{3}$.

8. **Answer: $\frac{5}{2}$**

   Suppose the medians intersect at P. If $\overline{BC} = x, \overline{BP} = \overline{CP} = \frac{x}{\sqrt{2}}$. By a well-known property of centroids, $\frac{MP}{MC} = \frac{1}{3}$, so $MP = \frac{x}{2\sqrt{2}}$. Using the Pythagorean Theorem, we find that $MB = \frac{x\sqrt{5/2}}{2} \Rightarrow AB = x \cdot \sqrt{\frac{5}{2}}$. So $\left(\frac{AB}{BC}\right)^2 = \frac{5}{2}$.

9. **Answer: $\mathbf{2\sqrt{3}}$**

   Stewart's Theorem states that $(PQ)^2(SR)+(QR)^2(PS)=(QS)^2(PR)+(PS)(PR)(SR)$. (This can be derived by applying the law of cosines to $\angle PSQ$ and $\angle RSQ$.) Plugging in and solving gives $QR=5/\sqrt{3}$, and applying the theorem again to triangle QST gives $ST=2\sqrt{3}$.

10. **Answer: $\frac{d\sqrt{3}}{3}+\frac{d}{2\pi}$**

    The car can drive the farthest (reaching Γ) by turning as sharply as possible for some distance then driving straight. Suppose it turns while driving a length s. Noting the correspondence between angle at the center of a circle and arclength, it has then reached the point $\left(\frac{d}{\pi},0\right)+\frac{d}{\pi}\left(-\cos\left(\frac{s\pi}{d}\right),\sin\left(\frac{s\pi}{d}\right)\right)$. It now has $d-s$ left to drive in the direction (since the tangent is perpendicular to the radius) $\left(\sin\left(\frac{s\pi}{d}\right),\cos\left(\frac{s\pi}{d}\right)\right)$. Adding up, we have a parametric equation for the boundary:

    $$\Gamma(s)=\left(\frac{d}{\pi},0\right)+\frac{d}{\pi}\left(-\cos\left(\frac{s\pi}{d}\right),\sin\left(\frac{s\pi}{d}\right)\right)+(d-s)\left(\sin\left(\frac{s\pi}{d}\right),\cos\left(\frac{s\pi}{d}\right)\right)$$

    Solving for s would be very difficult, but noting that one term has a π in the denominator and one doesn't, we can make the educated guess $s=\frac{d}{3}$; this indeed works.