Algebra Solutions
2007 Rice Math Tournament
February 24, 2007

1. **Answer: $\mathbf{4^{1/9}, -1}$**

   $f(x^{1/9}) = (x-4)(x+1)$ so $f=0$ means $x=4$ or $x=-1$, so $f(4^{1/9}) = f(-1) = 0$.

2. **Answer: $\mathbf{\frac{1}{4}}$**

   Since $(x-x_1)(x-4x_1) = x^2 - 5x_1x + 4x_1^2$, we know $3 = 2(4x_1^2 + 5x_1)$ so $x_1 = \frac{1}{4}$. (The other root is negative.)

3. **Answer: $\mathbf{-37}$**

   Simply note that $(a+b)(a+c)(b+c) = (ab+ac+bc)(a+b+c) - abc = -6 \cdot 7 + 5 = -37$.

4. **Answer: 1339**

   Let k be a nonnegative integer. Let $f(x) = \lfloor x \rfloor + \lfloor 2x \rfloor + \lfloor 3x \rfloor$. If $k \le x < k + \frac{1}{3}$, then $f(x) = 6k$. If $k + \frac{1}{3} \le x < k + \frac{1}{2}$, then $f(x) = 6k+1$. If $k + \frac{1}{2} \le x < k + \frac{2}{3}$, then $f(x) = 6k+2$. If $k + \frac{2}{3} \le x < k+1$, then $f(x) = 6k+3$. There is therefore only a solution if n is 0, 1, 2, or 3 mod 6; there are $2004 \cdot \frac{4}{6} + 3$ of these.

5. **Answer: $\mathbf{\frac{5}{144}}$**

   There are clearly five correct guesses; counting the number of possible guesses is the difficult part. A possible guess q is ± 1 times a divisor of 90 divided by a divisor of 400. We count these by extending the idea of prime factorization: from the factorizations of 90 and 400: we have $q = 2^i 3^j 5^k$ where $-4 \le i \le 1$, $0 \le j \le 2$, and $-2 \le k \le 1$. There are thus $6 \cdot 3 \cdot 4 = 72$ possible fractions making 144 possible guesses.

6. **Answer: 881**

   We can factor $4x^4 + y^4 = (4x^4 + 4x^2y^2 + y^4) - 4x^2y^2 = (2x^2+y^2)^2 - (2xy)^2 = (2x^2+2xy+y^2)(2x^2 - 2xy + y^2)$. Since $4^9 + 9^4 = 4(16)^4 + 9^4$, we plug in to obtain the factoring $881 \cdot 305$. Quick checking (up to 29) shows 881 to be prime.

7. **Answer: 6**

   The expression is 6 times the arithmetic mean of the terms, which is is always greater than or equal to the geometric mean, which is $xy \cdot x \cdot y \cdot \frac{1}{x} \cdot \frac{1}{y} \cdot \frac{1}{xy} = 1$. The minimum is achieved when all terms are equal, i.e. $x = y = 1$.

8. **Answer: 4**

   $(r+s+t)^3 - 3(r+s+t)(r^2+s^2+t^2) + 2(r^3+s^3+t^3) = 6rst$ - just plug in!

9. **Answer: $\mathbf{\sqrt{5}}$**

   Note that the equations reduce by substitution to $a = b + \frac{1}{a+1/a}$ and $b = a - \frac{1}{b+1/b}$. Solving the second for a, substituting into the first, and reducing yields $b^4 + b^2 - 1 = 0$; solving this as a quadratic in b^2 yields only one positive value for $b^2 = \frac{\sqrt{5}-1}{2}$. Plugging back in and solving for a gives $a^2 = \frac{\sqrt{5}+1}{2}$.

10. **Answer: $\mathbf{-2015028}$**

    Note that $(x+1)^2 - x^2 = 2x+1$ so:

$$
\begin{aligned}
\sum_{k=1}^{2007}(-1)^k k^2 &= -2007^2 + \sum_{k=1}^{1003}(2(2k-1)+1) \\
&= -2007^2 + 4\frac{1003\cdot 1003}{2} + 1003 \\
&= -2007^2 + 1003\cdot 2007 = 2007(1003-2007)
\end{aligned}
$$