Advanced Topics Solutions
2007 Rice Math Tournament
February 24, 2007

1. **Answer: 2**

   This gives three equations: $x+2y+2z=0$, $x+3y+4z=0$, and $3x+4y+kz=0$. To have a nonzero solution, the first two equations must agree with the last one, so we add 5 times the first to -2 times the second to get $3x+4y+2z$. Thus if $k=2$, we actually only have two equations and can get a nonzero solution.

2. **Answer: $\mathbf{\frac{1}{343}}$**

   We see that $\log_3 27 = \log_3 3^3 = 3\log_3 3 = 3$, and $\log_x 7 = \dfrac{1}{\log_7 x}$. Thus we have $3 = \log_{27} x \cdot \log_7 3 \cdot \log_7 x$. We also see that $\log_{27} x = \dfrac{1}{\log_x 3^3} = \dfrac{1}{3\log_x 3} = \dfrac{1}{3}\log_3 x$, giving us $9 = \log_3 x \cdot \log_7 3 \cdot \log_7 x$. Now we note that $\log_3 x \cdot \log_7 3 = \log_7 3 \cdot \log_3 x = \log_7 x$, so we have $9 = (\log_7 x)^2$, so $\log_7 x = \pm 3$, so $x = 7^{\pm 3} = 343, \frac{1}{343}$

3. **Answer: $\mathbf{\frac{10}{21}}$**

   By writing the number of ways to reach each vertex of the path, we see that we get Pascal's triangle, so the answer is $\frac{\binom{5}{3}\binom{4}{2}}{\binom{9}{5}} = \frac{10\cdot 6}{126} = \frac{10}{21}$

4. **Answer: 1024**

   By writing out the trustworthiness of the first several people, we see that it flips after each square; that is, if T represents a truth-teller and L represents a liar, we either have $L, T, T, T, L, L, L, L, L, T, \ldots$ or $T, L, L, L, T, T, T, T, T, L, \ldots$. For this to end up so that P_n is right about P_1, we must have n between an even square below and an odd one above, i.e. $(2k)^2 \le n < (2k+1)^2$ for some k, so the least such n is 1024.

5. **Answer: $\mathbf{2^n - 1}$**

   Let $g(n)$ denote the solution to the problem. Note that to move the bottom disk we must first move the top $n-1$, requiring $g(n-1)$ moves. We then can move the last disk to the desired peg, and then must move the other $n-1$ disks on top of it, taking another $g(n-1)$ moves. Clearly $g(1)=1$, and we have $g(n) = 2g(n-1)+1$, so $g(n) = 2^n - 1$. (We can check that this satisfies the recursive relation; the easiest way to guess it is simply to write out the first several values.)

6. **Answer: $\mathbf{-\frac{1}{16}}$**

   We have $x = r\cos\theta$.

$$\begin{aligned} x &= (\cos\theta + \frac{1}{2})\cos\theta = \cos^2\theta + \frac{1}{2}\cos\theta \\ &= \cos^2\theta + \frac{1}{2}\cos\theta + \frac{1}{16} - \frac{1}{16} \\ &= \left(\cos\theta + \frac{1}{4}\right)^2 - \frac{1}{16} \ge -\frac{1}{16} \end{aligned}$$

7. **Answer: $\boldsymbol{(k+1, n-k)}$**

   We consider adding n to a permutation of length $n-1$. Suppose we already have k ascents (there are $\left\langle {n-1 \atop k} \right\rangle$ such permutations); to add no ascents, n must either be placed at the beginning or in the middle of one of the k ascents, giving $a = k+1$. Now suppose we have $k-1$ ascents (there are $\left\langle {n-1 \atop k-1} \right\rangle$ such permutations); to add one ascent we must place n either at the end or after one of the $(n-1-1)-(k-1)$ descents, giving $b = n-k$.

8. **Answer: $n^2 + 2n$**

   We first note that this is clearly a lower bound, since if we are able to do this without ever moving backward, we must move each of the red pegs $n+1$ and each of the blue pegs $n+1$, but we save moves since each peg will jump or be jumped by the n pegs of the other color, making a total of $2(n+1)n - n^2 = n^2 + 2n$ moves. We see that we can achieve this lower bound by the following process. First we move the red pegs as far as we can without moving one next to another one, then the same for the blue pegs, then the red pegs again, and so on. (The first two steps move one red peg, then jump with a blue and move the next blue up). After doing this n times (taking $1+2+\cdots+n$ moves), the pegs will all be alternating, with the empty spot at one end. We can then make n consecutive jumps, ending with the empty spot at the other end, then repeat the first algorithm for another $n + (n-1) + \cdots + 1$ moves to be done. The total is then $2(n+1)n/2 + n = n^2 + 2n$.

9. **Answer: $p^2 + p + 2$**

   Let $a = b^2$ and let $X = \begin{pmatrix} u & v \\ y & z \end{pmatrix}$. The resulting equations are $u^2 + vy = yv + z^2 = a$ and $(u+z)v = (u+z)y = 0$. First suppose $u + z = 0$. The resulting solution is

   $$(u, v, y) = (\pm b, 0, 0); (\pm b, 0, t), (w, t, t^{-1}(b^2 - u^2))$$

   where $t \not\equiv 0 \bmod p$ and $w \not\equiv \pm b$. There are $2 + 2(p-1) + 2(p-1) + (p-2)(p-1) = p^2 + p$ solutions here. Otherwise $u + z \neq 0$ so $v = y = 0$ and $u^2 = z^2 = a$, giving $u, v = \pm b$, two more solutions.

10. **Answer: 1007**

    First note that $P(a) - P(b)$ is the sum of terms in the form $C_n \cdot (a^n - b^n) = C_n \cdot (a-b)(a^{n-1} + a^{n-2} \cdot b + \cdots + b^{n-1})$. So the sum is divisible by $(a-b)$ and by another complex sum that is certainly greater than 1 given that each C is a positive integer. So $P(a) - P(b)$ can only be prime if a and b differ by 1. Therefore, we cannot have more than 6 such pairs of elements in T. Clearly we could start with $1, 3, 5, \ldots, 2007$ and add in any three even numbers since each would produce 2 pairs of elements differing by 1.