Advanced Topics Test
2007 Rice Math Tournament
February 24, 2007

1. The equation $\begin{pmatrix} 1 & 2 & 2 \\ 1 & 3 & 4 \\ 3 & 4 & k \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ has a solution for (x, y, z) besides $(0, 0, 0)^t$. Find the value of k.

2. If $\log_3 27 \cdot \log_x 7 = \log_{27} x \cdot \log_7 3$, find the least possible x.

3. Scott had too much to drink at the math building last night. He came out of the math building at $(0, 0)$ facing in the positive y-direction. He walked to his home at $(4, 5)$ one block at a time, picking either the $+y$ or $+x$ direction. What is the probability that Scott passed Paula's house at $(2, 3)$ on the way home, assuming that all possible paths are equally likely? (Coordinates of points are given in units of blocks.)

4. A village has n residents, named P_1, P_2, $\cdots$, P_n. Each either tells the truth or lies all the time. For each k: If k is a perfect square, P_k says that P_{k+1} is lying. Otherwise, P_k says that P_{k+1} is telling the truth. (P_n talks about P_1.) What is the minimum number of residents, given that $n > 1000$?

5. The Tower of Hanoi game consists of three pegs, upon one of which are stacked n disks of radii $1, 2, \ldots, n$ from largest to smallest, bottom to top. The object is to move the stack to another peg by moving one disk at a time, never placing a disk on top of a smaller one. What is the minimum number of moves required to complete the game?

6. A graph is defined in polar coordinates by $r(\theta) = \cos\theta + \frac{1}{2}$. Find the smallest x-coordinate of any point on this graph.

7. A permutation $a_1, a_2, \ldots, a_n$ of the digits $1, 2, \ldots, n$ is said to have k ascents if for exactly k of the a_i, $a_i < a_i + 1$. Let $\left\langle {n \atop k} \right\rangle$ denote the number of perumtations of length n with k ascents. Find the ordered pair (a, b) such that $\left\langle {n \atop k} \right\rangle = a\left\langle {n-1 \atop k} \right\rangle + b\left\langle {n-1 \atop k-1} \right\rangle$.

8. A board has $2n+1$ holes in a line, with n red pegs starting in the first n holes, followed by a gap, then n blue pegs in the remaining n holes. The pegs may be moved either one hole forward into an empty hole or a peg of one color may jump over a peg of the other color into an empty hole on the other side. What is the fewest number of moves it takes to completely interchange the red and blue pegs?

9. Let p be an odd prime, and let a be the square of a non-zero element mod p. Find the number of 2×2 matrices X such that $X^2 \equiv \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}$ mod p.

10. Let S be the set $1, 2, \ldots, 2007$, and let P be any polynomial of degree 2007 with positive integer coefficients. A subset T of S is chosen so that the set $\{|P(a) - P(b)| : a \neq b, a, b \in T\}$ cannot contain more than 6 prime numbers. What is the largest possible number of elements T can contain?