Team Test
2006 Rice Math Tournament
February 25, 2006

1. Given $\triangle ABC$, where A is at $(0,0)$, B is at $(20,0)$, and C is on the positive y-axis. Cone M is formed when $\triangle ABC$ is rotated about the x-axis, and cone N is formed when $\triangle ABC$ is rotated about the y-axis. If the volume of cone M minus the volume of cone N is 140π, find the length of $\overline{BC}$.

2. In a given sequence $\{S_1, S_2, \ldots, S_k\}$, for terms $n \geq 3$, $S_n = \sum_{i=1}^{n-1} i \cdot S_{n-i}$. For example, if the first two elements are 2 and 3, respectively, the third entry would be $1 \cdot 3 + 2 \cdot 2 = 7$, and the fourth would be $1 \cdot 7 + 2 \cdot 3 + 3 \cdot 2 = 19$, and so on . Given that a sequence of integers having this form starts with 2, and the 7th element is 68, what is the second element?

3. A triangle has altitudes of lengths 5 and 7. What is the maximum possible integer length of the third altitude? *(We restricted the third altitude to integer lengths after the contest)*

4. Let $x + y = a$ and $xy = b$. The expression $x^6 + y^6$ can be written as a polynomial in terms of a and b. What is this polynomial?

5. There exist two positive numbers x such that $\sin(\arccos(\tan(\arcsin x))) = x$. Find the product of the two possible x.

6. The expression $16^n + 4^n + 1$ is equivalent to the expression $(2^{p(n)} - 1)/(2^{q(n)} - 1)$ for all positive integers $n > 1$ where $p(n)$ and $q(n)$ are functions and $\frac{p(n)}{q(n)}$ is constant. Find $p(2006) - q(2006)$.

7. Let S be the set of all 3-tuples (a,b,c) that satisfy $a + b + c = 3000$ and $a, b, c, > 0$. If one of these 3-tuples is chosen at random, what's the probability that a, b, or c is greater than or equal to $2,500$?

8. Evaluate: $\lim_{n\to\infty} \sum_{k=n^2}^{(n+1)^2} \frac{1}{\sqrt{k}}$

9. $\triangle ABC$ has $AB = AC$. Points M and N are midpoints of $\overline{AB}$ and $\overline{AC}$, respectively. The medians $\overline{MC}$ and $\overline{NB}$ intersect at a right angle. Find $\left(\frac{AB}{BC}\right)^2$.

10. Find the smallest integer $m > 8$ for which there are at least eleven even and eleven odd positive integers n so that $\frac{n^3+m}{n+2}$ is an integer. *(We restricted the solution to $m > 8$ after the contest since $m = 8$ is a trivial solution, with $n^3 + 8$ divisible by $n + 2$)*

11. Polynomial $P(x) = c_{2006}x^{2006} + c_{2005}x^{2005} + \ldots + c_1 x + c_0$ has roots $r_1, r_2, \ldots, r_{2006}$. The coefficients satisfy $2i\frac{c_i}{c_{2006-i}} = 2j\frac{c_j}{c_{2006-j}}$ for all pairs of integers $0 \leq i, j \leq 2006$. Given that $\sum_{i\neq j, i=1, j=1}^{2006} \frac{r_i}{r_j} = 42$, determine $\sum_{i=1}^{2006}(r_1 + r_2 + \ldots + r_{2006})$.

12. Find the total number of k-tuples $(n_1, n_2, \ldots, n_k)$ of positive integers so that $n_{i+1} \geq n_i$ for each i, and k regular polygons with numbers of sides $n_1, n_2, \ldots, n_k$ respectively will fit into a tesselation at a point. That is, the sum of one interior angle from each of the polygons is $360°$.

13. A ray is drawn from the origin tangent to the graph of the upper part of the hyperbola $y^2 = x^2 - x + 1$ in the first quadrant. This ray makes an angle of θ with the positive x-axis. Compute $\cos\theta$.

14. Find the smallest nonnegative integer n for which $\binom{2006}{n}$ is divisible by 7^3.

15. Let c_i denote the ith composite integer so that $\{c_i\} = 4, 6, 8, 9, ...$ Compute

$$\prod_{i=1}^{\infty} \frac{c_i^2}{c_i^2 - 1}.$$

(Hint: $\sum_{i=1}^{n} \frac{1}{n^2} = \frac{\pi^2}{6}$).