Power Solutions
2006 Rice Math Tournament
February 25, 2006

1. **Answer: *Blue*, *Red*, *Red*, *Blue***

   We see that in all of the examples, even those with straight and wavy segments connected, neither player will ever be able to remove any of the other player's edges, so long as the players play optimally and do not detach their own segments from the ground. Therefore, the players will simply take turns removing segments, and the winner will be the player with more segments to begin with.

2. The second player has a winning strategy: Every time the first player moves, the second player can simply remove the corresponding segment. This ensures that the second player always has a move, and the first player therefore loses. This strategy will work for all games consisting of two games side-by-side where each is the same as the other with straight segments replaced by wavy and vice versa. It will also work for similarly constructed games in which the two halves are connected/overlapping. Note that this strategy does not necessarily lead to a win in the fewest possible moves.

3. **Answer: $(3, -3, -1, 0); 0$** Let a simple game of value n be n straight segments if $n > 0$ and n wavy segments if $n < 0$. The definition of value qualitatively says that straight edges are worth $+1$ and wavy edges are worth -1, so for each of the games in problem 1, we intuitively guess that the value is $s - w$, the number of straight segments minus the number of wavy segments. To prove this, we add a simple game of value $w - s$. The first player always loses in these games since the players take turns removing segments and always end up after some even number of turns with no segments left. Thus these games have value 0, and our guess was correct. In problem 2, we found a strategy that ensured that the first player always lost so the value of any of the games the strategy applies to is by definition 0.

4. Without loss of generality say $v > 0$ since the proof for $v < 0$ must be the same with Blue and Red's segment's interchanged. Since v is rational, for some n, n copies of G (call this nG will have integer value $k > 0$. Thus nG placed next to a game consisting of k wavy edges, the new game will have value zero, and by definition the first player loses. Therefore if we remove at least one of the wavy segments, Blue will win, and so Blue wins the game nG. Now suppose Blue cannot win the game G. This means that there is some strategy for Red that will ensure that Red wins G. Red can simply apply this strategy to whichever copy of G in nG Blue just moved in (if Red is first to move, Red can move in any one of the copies), and therefore will ensure that Blue loses in all copies, and thus the entire game nG. This is a contradiction, so Blue must be able to win the game G.

5. Suppose two possible moves for Blue leave G_1 and G_2 with values $v_1 < v_2$. Invert G_1 by replacing all straight segments with wavy ones and vice versa, then add it to G_2. The value is therefore still positive and in favor of Blue. Thus G_2 is stronger for Blue than Red's version of G_1 would be for Red, and so G_2 is the better move for Blue. Note that the proof also implies that if two moves leave the same value they are equally good.

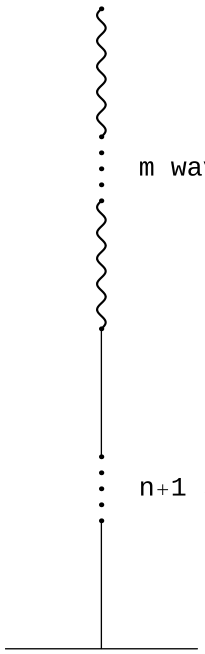
6. In each part we check that a game has zero value, that is, that the first player always loses. Often the results of problems 4 and 5 are useful, though in some cases trial and error is necessary.

   a. Add two copies of this game to a game with one wavy segment to make a zero game. The value satisfies $v + v - 1 = 0$.

   b. Add two copies of this game to a game of value $-\frac{1}{2}$, the inverse of the one from part a to make a zero game.

   c. Add this game to one with value $-\frac{1}{2}$ as in part b and a simple game with value -1 to make a zero game.

7. We can split the shown game into a simple game of value n and another game of unknown value by taking the top straight segment and all segments above it off the stack and attaching them to the ground.

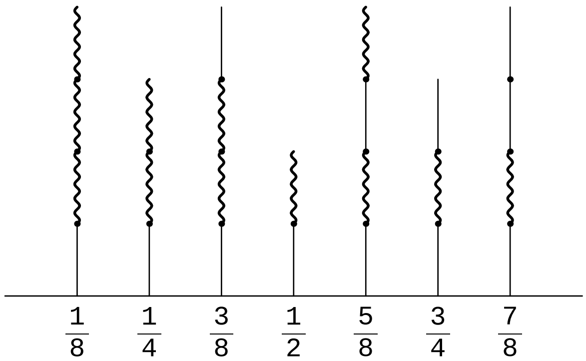


This does not change the game since the straight edges moved could not be affected by Red and would never be removed by Blue without first removing the unmoved straight edge. Now we must show that the straight edge with n wavy edges has value $\frac{1}{2^n}$. We therefore copy it 2^n times and add it to a single wavy segment. Using strong induction, we can assume that the values for m wavy segments, $m < n$ are $\frac{1}{2^m}$ (we already have the base case from the previous problem). It is also clear that adding another wavy segment must decrease the value. It is therefore favorable for Blue to remove the copy with the most wavy segments left on top of it. Similarly Red should always remove a wavy segment off the tallest stack. Therefore after 2^{n-1} moves by each player, the game is left with 2^{n-1} copies of the game, each with one segment removed from the top, along with the single added wavy segment. Using our inductive hypothesis, this game has zero value, so the first player to move in it loses. An even number of moves have been made up to this point, so the first player to move in this game was the first one to move orginially. Therefore the game we constructed has zero value, and so the value of a single stack of one straight and n wavy segments has value $\frac{1}{2^n}$.

8. Notice that the $\{p|q\}$ notation is well-defined; that is, all games in which Blue's best move is to p and Red's best move is to q have the same value (by arguments similar to problems 4 and 5); therefore we merely need to find an example of this identity in order to confirm it for all games with the given values. Examining our solution to problem 7, we see that $\frac{1}{2^{n+1}} = \{0|\frac{1}{2^n}\}$. The steps in which sums are split and combined are valid since there is a game consisting of a sum of games of value $\frac{1}{2^{n+1}}$ which works as an example.

   Then
$$\frac{2p+1}{2^{n+1}} = \frac{p}{2^n} + \frac{1}{2^{n+1}} = \frac{p}{2^n} + \{0|\frac{1}{2^n}\} = \{\frac{p}{2^n}|\frac{p+1}{2^n}\}$$

9. The values of the games are simple to determine from problem 8 and other previous results. Notice that the pattern can easily be extended to $\frac{k}{16}$, $\frac{k}{32}$, and so on, creating a base two decimal number line!



10. In the game below, the first player always wins, so it has no advantage for Blue or Red on its own, so we would expect that it might have zero value. However, if we place another copy next to it, Blue always wins, since there are more straight segments than wavy, so Blue can remove these until Red is forced to remove a dashed edge, at which point Blue takes the other one and wins. It then must have a positive value. But if we place a game of value $\frac{1}{2^n}$ next to it, Blue always wins, and similarly adding it to a game of value $-\frac{1}{2^n}$ lets Red always win. The value is therefore less than all positive numbers and greater than all negative numbers, but not zero!