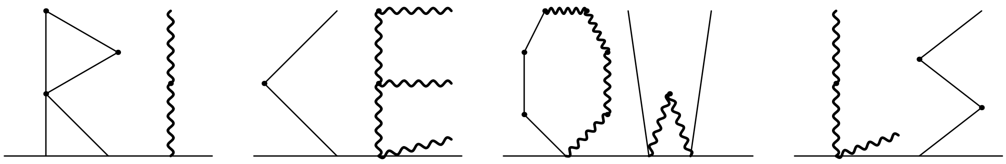
Power Test
2006 Rice Math Tournament
February 25, 2006
idea from Winning Ways for Your Mathematical Plays by Berlekamp, Conway, and Guy

Hackenbush is a game for two players, Blue and Red. It is played with a picture such as the ones in the first two problems, consisting of straight and wavy edges connected to each other and to the ground. Blue moves by deleting any *straight* line segment, together with any segments that are no longer connected to the ground. Red moves by similarly deleting a *wavy* segment. If a player has no edges left to delete, that player loses. Both players always make the best possible moves.

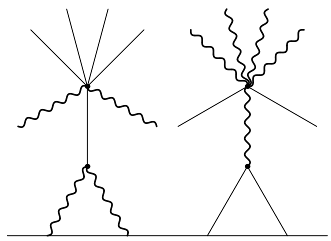
We have used straight and wavy edges on this test, but we request that in your solutions you use red pen (we have provided some) for Red's edges, and blue/black pen or pencil for Blue's edges.

1. Who wins in each of the following games?



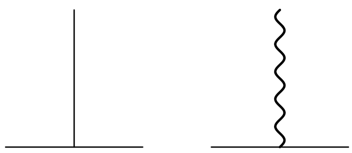
(a) (b) (c) (d)

2. Find a winning strategy for one of the players in the following game. For what other games will your strategy ensure a win for one of the players?



Let the value of a game be a number associated with the game so that the following properties hold:

- These games have value 1 and -1, respectively.



- The value of a game in which the first player to move always loses (with both players playing optimally) is zero. This is called a **zero game**.
- For two games G and H with values g and h placed side-by-side, the value of the whole game, $G+H$, is $g+h$.

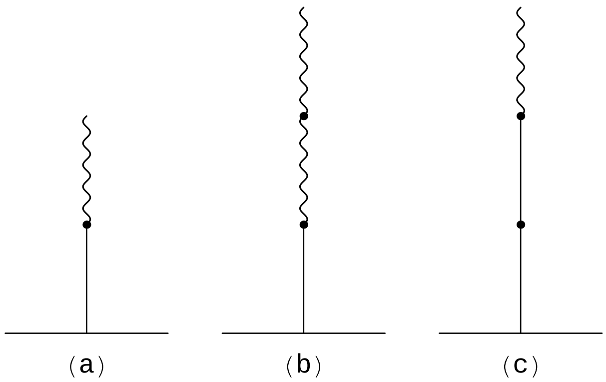
Notes: You may assume that these properties are consistent. We say a game has value v or is worth v moves.

3. Find the value of each of the games shown in problems 1 and 2. *We miscounted numbers of edges in the games in problem 1 in our original solutions, and also had problems with the sizes of dots separating segments. Both issues have been resolved here.*

4. Prove that that if the value v of a game G is a rational number, Blue wins if $v > 0$ and Red wins if $v < 0$.

5. Prove that the best move for Blue is the one that results in a game with the greatest value, and similarly the best move for Red is the one that leaves the least value.

6. Determine with proof the value of the following three games.



Notice that in the first game in problem 6, Blue's best move is to a game worth 0 moves, while Red's best move is to a game worth 1 move. We denote this $\{0|1\} = v$, where v is the value of this game.

7. Construct with proof games with values $n + \frac{1}{2^m}$, where n, m are integers, $m \geq 0$. Your solution should be a single vertical line like the examples in problem 6; that is, only one edge should connect to the ground and only two edges should meet at any point.

8. Prove that $\frac{2p+1}{2^{n+1}} = \{\frac{p}{2^n}|\frac{p+1}{2^n}\}$, where p, n are integers with $n \geq 0$.

9. Construct with proof games of values $\frac{1}{8}, \frac{2}{8}, \cdots, \frac{7}{8}$. Your solutions should be single vertical lines as in problem 7.

10. Show that if **thick** edges, which both Blue and Red may remove, are introduced, then there are games whose value is undefined.