Team Solutions
2005 Rice Math Tournament
February 26, 2005

1. **Answer: 2003**

   2003 appears in 2003!, 2004! and 2005!.

2. **Answer: 72**

   The interior angles of a regular polygon are always less than 180^o in measure, and the size of the angles increase with the number of sides. Therefore, if a polygon has angles that measure 175^o, it would be the solution since 175 is the largest multiple of 7 less than 180. Using the formula for interior angles: $\frac{180(n-2)}{n} = 175$; $5n = 360$; $n = 72$ So a 72-gon does in fact have angles of 175^o.

3. **Answer:** $\boldsymbol{\frac{7}{72}\pi h^3}$

   We can calculate the volume by adding up the volume of the frushrums. Since the base radius halves with each cone, and the height stays constant, the volume quarters for each successive frustrum. The volume of the first is $\frac{1}{3}\pi(\frac{h}{2})^2h - \frac{1}{3}\pi(\frac{h}{4})^2\frac{h}{2} = \frac{7}{96}\pi h^3$. The infinite sum is $\frac{v_0}{1-r} = \frac{\frac{7}{96}\pi h^3}{1-\frac{1}{4}} = \frac{7}{72}\pi h^3$.

4. **Answer:** $\boldsymbol{\frac{13}{6}}$

   The trick is to examine the area of the triangles APR, CRQ and BQP. The ratio of the area of $\triangle APR$ to that of $\triangle ARB$ is $\frac{1}{1+r}$, and the ratio of the area of $\triangle ARB$ to that of $\triangle ABC$ is $\frac{r}{r+1}$, so the area of $\triangle APR$ is $\frac{25r}{(r+1)^2}$. The others have the same area, so the total area of $\triangle ABC$ not including $\triangle PQR$ is $\frac{75r}{(r+1)^2}$. This equals $25 - 7 = 18$. Its not hard to deduce that $r^2\frac{13}{6}r + 1 = 0$, so the sum of the values of r is $\frac{13}{6}$. You can also check that they are both real ($\frac{3}{2}$ and $\frac{2}{3}$).

5. **Answer:** $\boldsymbol{50\sqrt{10203}}$

   Suppose that facing the first flight of stairs is North. Then every 4 floors brings you 2ft West and 2ft South, by the end of 100 floors, you are $\sum_{i=1}^{100} i = 50 \times 101ft$ high. The total distance is $\sqrt{50^2 + 50^2 + (50 \times 101)^2} = 50\sqrt{1 + 1 + 10201} = 50\sqrt{10203}$

6. **Answer: 9**

   If a coordinate has the same parity for two separate lattice points, then the midpoint will have an integer in that coordinate. So for all pairs of lattice points on the list, we have to make sure at least one of the parities of the three coordinates is different. That gives us 2 choices for the first coordinate, 2 for the second, and 2 for the third. Once we have all 8 possibilities, the 9th will guarrantee that it will have the same parity in all 3 coordinates as some a_i already on the list. Thus the answer is 9.

7. **Answer:** $\boldsymbol{3, 4, 5, 7}$

   If the probability of a sum is P, the probability of winning on repeat rolls with that sum is $\frac{P}{P+\frac{8}{96}} \times P$. Also, the total probability of the given winning rolls is $\frac{15}{96}$. Therefore, we have

   $$\left(\sum P_i \frac{P_i}{P_i + \frac{8}{96}}\right) + \frac{15}{96} = \frac{23242}{110880}$$

   Multiply through by 96 to simplify:

   $$\left(\sum (96P_i) \frac{P_i}{P_i + \frac{8}{96}}\right) + 15 = \frac{23242}{1155}$$

Calculating the possible values of $96P_i\frac{P_i}{P_i+\frac{8}{96}}$ we find: (excluding repetitions)

$$3\left(\frac{2}{5}\right),4\left(\frac{9}{11}\right),5\left(\frac{4}{3}\right),7\left(\frac{18}{7}\right),8\left(\frac{49}{15}\right),9\,(4)$$

Noting that $1155 = 3\times 5\times 7\times 11$ and that the fractional part of the expression comes from a sum of $96P_i\frac{P_i}{P_i+\frac{8}{96}}$, we check $\frac{2}{5}+\frac{9}{11}+\frac{4}{3}+\frac{18}{7}+15=\frac{23242}{1155}$, and so the 4 sums are $\{3,4,5,7\}$.

8. **Answer: 13**

   If h and s represent the hypotenuse and the remaining side lengths respectively, it follows from the Pythagorean Theorem that $(h-s)(h+s) = 3600 = 2^4 3^2 5^2$. For h and s to be integers, their sum and difference must both be even and positive, and $h+s > h-s$. Since 3600 has $(5)(3)(3) = 45$ factors, it is easy to see that there are 22 pairs $(h+s, h-s)$ with both terms positive and $h+s > h-s$. All ordered pairs have even elements except those with a factor of 225. There are 9 of these and each ordered pair pertains to only one set of values for h and s, so the answer is $22-9 = 13$.

9. **Answer: $\frac{31}{511}$**

   Any set of n elements has $2n$ subsets because each element of the set may or may not be included in a given subset. So we have 512 subsets of S, but one is empty, so there are 511 total nonempty subsets of S. For a subset of S to be reflective, it must contain both m and $10-m$ or contain neither m nor $10-m$ for $m = 1,2,\dots,9$. So each of the following sets must be subsets of or disjoint with any reflexive subset of $S: 1,9,2,8,3,7,4,6,5$. So there are $25-1 = 31$ such reflexive sets. The answer is $\frac{31}{511}$.

10. **Answer: 2005.0**

$$2000\cdot 2010 = (2005-5)(2005+5) = 2005^2 - 5^2 = 2005^2(1-\frac{5^2}{2005^2}) = 2005^2(1-\frac{1}{401^2}).$$

Hence,

$$\sqrt{2000\cdot 2010} = \sqrt{2005^2(1-\frac{1}{401^2})} = 2005\cdot\sqrt{1-\frac{1}{401^2}}.$$

Using the Taylor approximation for $f(x) = \sqrt{1-x^2}$ around $x = 0$,

$$f(x) \approx f(0) + f'(0)x + \frac{f''(0)x^2}{2}$$

$$f' = \frac{-x}{\sqrt{1-x^2}}$$

$$f'' = \frac{-1}{\sqrt{1-x^2}} + x^2\cdot(1-x^2)^{-\frac{3}{2}}.$$

Hence,

$$f(\frac{1}{401}) \approx 1 - \frac{1}{2\cdot 401^2}$$

$$2005\cdot(1-\frac{1}{2\cdot 401^2}) \approx 2005\cdot(1-\frac{1}{300,000}) \approx 2005.0\ (\textit{nearest tenth})$$

Note: hundredth is harder ≈ 2004.99.

11. **Answer: $\frac{5}{128}$**

    Consider the figure to be an overlap of 3 triangles with side length $2x$. Call these 3 A, B, and C. Then let $P(A)$ be the likelihood that A is entirely blue, and likewise for B and C. We need $P(A\cup B\cup C) = P(A)+P(B)+P(C)-P(A\cap B)-P(B\cap C)-P(A\cap C)+P(A\cap B\cap C)$. $P(A) = P(B) = P(C)) = \frac{2^5}{2^9}$. $P(A\cap B)$ and symmetric values are $\frac{2^2}{2^9}$. $P(A\cap B\cap C) = 0$. The answer is $\frac{2^5-3\cdot 2^2}{2^9} = \frac{5}{128}$.

12. **Answer: $180 - 45\sqrt{2} - 30\sqrt{3} - 15\sqrt{6}$**

In the "x" component, he travelled $30(\cos 0 + \cos 15 + \cos 30 + \cos 45 + \cos 60 + \cos 75 + \cos 90)$, and the same in the "$y$" direction. This is $30(1 + \cos 15 + \frac{\sqrt{3}}{2} + \frac{\sqrt{2}}{2} + \frac{1}{2} + \sin 15$. Then $\cos 15 + \sin 15 = (\sqrt{2})(\cos 45 \sin 15 + \sin 45 \cos 15) = \sqrt{2} \sin 60 = \frac{\sqrt{6}}{2}$. So the distance in the x or y direction is $30(\frac{3+\sqrt{2}+\sqrt{3}+\sqrt{6}}{2})$. So the total distance is $15(2 + 3\sqrt{2} + \sqrt{6} + 2\sqrt{3})$ and the answer is $210 - (30 + 45\sqrt{2} + 30\sqrt{3} + 15\sqrt{6}) = 180 - 45\sqrt{2} - 30\sqrt{3} - 15\sqrt{6}$.

13. **Answer: 4**

Recall $\sin 2x = 2 \cos x \sin x$.

$$2^{999}\left(\sin \frac{\pi}{2^{1000}}\right)P = \sin \frac{\pi}{2} = 1.$$

$$P \approx \frac{1}{2^{999} \sin \frac{\pi}{2^{1000}}}$$

But for $x \sim 0, \sin x \sim x \to \frac{1}{2^{999} \sin \frac{\pi}{2^1000}} \sim \frac{2^{1000}}{2^{999}\pi} = \frac{2}{\pi}$.

The answer is approximately $2\pi \cdot \frac{2}{\pi} = 4$.

14. **Answer: $\frac{1}{2}$**

Note that $f(x) = \frac{1}{x-i} = \left(\frac{x}{x^2+1}\right) + \left(\frac{1}{x^2+1}\right)i$

Let $T = \frac{1}{x^2+1} \Rightarrow x = \sqrt{\frac{1}{T} - 1}$

Then $f(x) = T\sqrt{\frac{1}{T} - 1} + Ti = \sqrt{\frac{1}{4} - (T - \frac{1}{2})^2} + Ti$.

Writing this as $A + Bi$, we see that

$$A^2 + \left(B - \frac{1}{2}\right)^2 = \frac{1}{4} = \left(\frac{1}{2}\right)^2$$

Thus $f(x)$ is a circle in the complex plane contered at $(0, \frac{1}{2}i)$ with radius $\frac{1}{2}$. For a square centered in the circle, the side length is $\frac{1}{\sqrt{2}}$. The area is $\frac{1}{2}$.

Alternatively, we can see that the image of f is the circle $u^2 + v^2 = v$ of radius 1/2 centered at $(0, \frac{1}{2})$. Hence the area of an inscribed square is $\frac{1}{2}$.

15. **Answer: $-\frac{2}{5}$**

Let

$$S = \sum_{n=0}^{\infty} (\frac{-1}{2})^n \cdot F_n.$$

$$S = F_0 - \frac{1}{2}F_1 + \frac{1}{4}F_2 - \cdots$$

Then

$$S = (F_0 - \frac{1}{2}F_1 + \frac{1}{8}F_2) + (\frac{1}{8}F_2 - \frac{1}{16}F_3) - (\frac{1}{16}F_3 - \frac{1}{32}F_4) + \cdots + (\frac{-1}{2})^n \cdot \frac{1}{2} \cdot (F_n - \frac{1}{2}F_{n+1}) + \cdots$$

Observe

$$F_n - \frac{1}{2}F_{n+1} = \frac{1}{2}F_n + (\frac{1}{2}F_n - \frac{1}{2}F_{n+1})$$

$$\frac{1}{2}F_n - \frac{1}{2}F_{n-1}$$

$$\frac{1}{2}F_{n-2}$$

So

$$S = (F_0 - \frac{1}{2}F_1 + \frac{1}{8}F_2) + \sum_{n=2}^{\infty}(-\frac{1}{2})^n \cdot \frac{1}{2} \cdot \frac{1}{2}F_{n-2}$$

$$S = (F_0 - \frac{1}{2}F_1 + \frac{1}{8}F_2) + \frac{1}{16}\sum_{n=2}^{\infty}(-\frac{1}{2})^{n-2}F_{n-2}$$

$$S = (F_0 - \frac{1}{2}F_1 + \frac{1}{8}F_2) + \frac{1}{16}S$$

Hence

$$S - \frac{1}{10}S = F_0 - \frac{1}{2}F_1 + \frac{1}{8}F_2$$

$$S - \frac{1}{10}S = -\frac{1}{2} + \frac{1}{8}$$

$$S - \frac{1}{10}S = -\frac{3}{8}$$

$$\frac{15}{16}S = -\frac{3}{8}$$

So

$$S = -\frac{2}{5}.$$