Advanced Test
2005 Rice Math Tournament
February 26, 2005

1. Billy Bunny is a compulsive liar. He lies $\frac{2}{5}$ of the time. However, a clue to his validity is that his ears droop $\frac{2}{3}$ of the time when he is telling a lie. They only droop $\frac{1}{10}$ of the time when he is telling the truth. Billy Bunny told me, "I love the Rice Math Tournament", but I noticed that his ears were drooping. What is the probability that he was telling the truth?

2. Find the last 4 digits of 2005^{2005}.

3. A toroidal chess board is like an ordinary chess board except that pieces can move off the edges and appear on the other side. A knight can attack by moving 2 squares in one direction and one square in a perpendicular direction. For instance, the knight at A can attack any of the starred squares:

*		*					
*		*					
			*				*
	A						
			*				*

Four knights are placed at random on a toroidal chess board. What is the probability that none of them can attack each other?

4. The solutions to $z^6 = 1$ and $z^4 = 1$ each create the vertices of a polygon in the complex plane. Find the area of the union of these two polygons.

5. In how many zeroes does 2005! end?

6. Evaluate

$$\frac{2005}{2005 + \frac{2005}{2005 + \frac{2005}{2005 + \cdots}}}.$$

The solution should be of the form: $\frac{\sqrt{(A)(B)} - A}{2}$.

7. Bessie the spherical cow offers you a bet. She has k distinct integers written on slips of paper in her shoe. If you take the bet, Bessie will choose one of these integers randomly. If the integer she chooses is n, then she will place the integers 1 through n in her hat. You will then choose one of these n integers and will win three muffins if your integer is relatively prime to n. Otherwise you give Bessie a muffin. If there are four winning numbers in the hat, regardless of which number is removed from the shoe, what is the expected number of muffins you will win or lose each game?

8. Let P be a polynomial with positive integer coefficients and degree at least 2. If x and y are randomly chosen among $1, 2, 3, 4, ..., 40$ with $x > y$, what is the probability that $|P(x) - P(y)|$ is necessarily composite?

9. Let $S_{m,k}$ be a set of k (k is an integer) consecutive integers whose least element is the positive integer m. Find all sets $S_{m,k}$ such that

$$\sum_{n \in S_{m,k}} n = 2005.$$

10. Find all ordered pairs of positive integers (m, n) where $m < 10$, and $5 < n < 15$ such that $5^m + 3^n - 1$ is divisible by 15.