Power Solutions
2004 Rice Math Tournament
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The material for this test was based on a theorem by Larson and Dai discovered in 1997.

1. $[-\frac{32\pi}{7}, -4\pi)+6\pi \cup [-\pi, -\frac{4\pi}{7})+2\pi \cup [\frac{4\pi}{7}, \pi) \cup [4\pi, \frac{32\pi}{7})-4\pi = [\frac{10\pi}{7}, 2\pi) \cup [\pi, \frac{10\pi}{7}) \cup [\frac{4\pi}{7}, \pi) \cup [0, \frac{4\pi}{7}) = [0, 2\pi)$.
2. $[-\frac{32\pi}{7}, -4\pi)*2^{-2} \cup [-\pi, -\frac{4\pi}{7})*2 \cup [\frac{4\pi}{7}, \pi)*2 \cup [4\pi, \frac{32\pi}{7})*2^{-2} = [-\frac{8\pi}{7}, -\pi) \cup [-2\pi, -\frac{8\pi}{7}) \cup [\frac{8\pi}{7}, 2\pi) \cup [\pi, \frac{8\pi}{7}) = I_2$
3. $[-\pi, -\frac{\pi}{2}) * 2 \cup [\frac{3\pi}{2}, 2\pi) \cup [2\pi, 3\pi) * 2^{-1} = [-2\pi, -\pi) \cup [\pi, 2\pi)$.

   $[-\frac{\pi}{2}, -\frac{\pi}{4}) + 2\pi \cup [\frac{7\pi}{4}, 2\pi) \cup [2\pi, \frac{7\pi}{2}) - 2\pi = [0, 2\pi)$.
4. Assume 0 is an element of E. Then because $E\ {}_2I_2$, some subset of E containing 0 must be multiplied by a power of 2 to produce a subset of $[-2\pi, -\pi) \cup [\pi, 2\pi)$. Obviously zero cannot be multiplied by any power of 2 to get a value in this set. So E is not 2-dilation congruent to I_2, and is thus not a wavelet set.
5. If $[a, b) \sim_{2\pi} [0, 2\pi)$, then we can split the interval $[a, b)$ into 2 intervals: $[a, 2n\pi) \cup [2n\pi, b)$ (as long as $a \neq 2n\pi$ for some n, in which case, $b = 2(n+1)\pi$). Then $a - 2(n-1)\pi = b - 2n\pi$ when we shift $2n\pi$ to 2π and 0. Note these must meet exactly. If they do not, either there will be overlap or a gap in $[0, 2\pi)$. So $b - a = 2\pi$.

   If $b - a = 2\pi$, one can use the exact procedure above to break the interval and show that $[a, b) \sim_{2\pi} I_1$.
6. The reasoning in this is similar to that in number 5. If $[a, b) \sim_2 [\pi, 2\pi)$, then we can write $[a, b)$ as $[a, 2^j\pi) \cup [2^j\pi, b)$. Then dividing by 2^j and 2^{j-1} gives$a * 2^{-(j-1)} = b * 2^j$. So $b = 2a$.

   If $b = 2a$, follow the same process to break up the interval to show $[a, b) \sim_2 [\pi, 2\pi)$.
7. Because $E \sim_2 I_2$, E must have a subset that is positive for $[\pi, 2\pi)$ and a subset that is negative for $[-2\pi, -\pi)$. Since E has only 1 interval, it must contain 0. But from (4), 0 cannot be an element of a wavelet set. Thus there cannot be any 1 interval wavelet sets.
8. Because E is 2-dilation congruent to $[-2\pi, -\pi) \cup [\pi, 2\pi)$ and we can only multiply the subsets by powers of 2 (which will be positive), c has to be positive and b has to be negative. So $[c, d)$ is a positive interval that must be 2-dilation congruent to $[\pi, 2\pi)$. Thus $d = 2c$. By the same token, $[a, b)$ is a negative interval that must be 2-dilation congruent to $[-2\pi, -\pi)$. So $a = 2b$.
9. We start from $[2b, b) \cup [c, 2c)$. Now for this set to be 2π-translation congruent to $[0, 2\pi)$, $b + 2n\pi = c$, so that the sets meet in $[0, 2\pi)$. So, $E \sim_{2\pi} [2b + 2n\pi, c) \cup [c, 2c)$. Thus, we have one interval that from (5) above must have $2c - c + b - 2b = 2\pi$. Thus $c - b = 2\pi$. So for this to be a wavelet set, we have $[2b, b) \cup [2b + 2\pi, b + 4\pi)$. Note that for this to make sense, $-2\pi < b < 0$.
10. We notice that one interval must be negative and the other two positive. So from above, make $u = 2v$. Then $[x, y) \cup [w, z) \sim_{2\pi} [2v + 2\pi, v + 4\pi)$ and it is 2-dilation congruent to $[\pi, 2\pi)$ and thus to $[2v + 2\pi, v + 4\pi)$. (also, if $b = -v$, we have $[-2b + 2\pi, -b + 4\pi)$)
11. This is the (only) negative interval. Suppose b is not less than π. Then $2b \geq 2\pi$. When we make E 2π-translation congruent to $[0, 2\pi)$, we will get $[0, 2\pi - 2b) \cup [4\pi - b, 2\pi)$ as the contribution from $[-2b, -b)$. Now we want to break the one interval $[2\pi - 2b, 4\pi - b)$ into 2 intervals. Because $[0, 2\pi - 2b)$ is definitely covered by the negative interval, we must make the 2 positive intervals have values larger than $2\pi - 2b$. To break this into 2 intervals, we can only move part of the interval by positive multiples of 2π. We must keep the length of this segment that we move constant, and we must keep its length positive, or else there are only 2 intervals. Unfortunately, when we shift to the right, we must have

$[x, y) \cup [w, z)$ 2-dilation congruent to the original interval. Thus we must multiply $[w, z)$ by powers of $\frac{1}{2}$ to get it back into the original interval. This makes $[w, z)$ shorter by a factor of 2^{-j} and it no longer will line up with the original interval. Note that taking out 2 intervals and shifting them by $2n\pi$ only makes things worse when we have to multiply both intervals by 2^{-j}. So what this shows us is we need a little "space" around 0 to make sure we can shift one interval to the left and one to the right. Thus $2b < 2\pi$ and $b < \pi$.

12. In this step, we use 2π-translation. Our interval $[-2b, -b)$ shifts to $[2\pi - 2b, 2\pi - b)$ and this interval is a subset of $(0, 2\pi)$ because $0 < b < \pi$. We note that our 2 intervals must have endpoints that line up with $2\pi - b$ and $4\pi - 2b$. We also want one positive interval to be shifted by a positive multiple of 2π to fit in $[2\pi - b, 4\pi - 2b)$. Our first 2 options to the left of $2\pi - b$ are $[x, 2\pi - 2b)$ and $[-b, x)$. The second is already negative, so our only choice has to be $[x, 2\pi - 2b)$. Then for our other interval, it must be 2π-translation congruent to $[2\pi - b, 2\pi + x)$, and we want to shift it by negative multiples of 2π to obtain this interval. So we can write the third interval as $[2n\pi - b, 2n\pi + x)$, for some $m > 1$ and some $x > 0$.

13. Now we want to use 2-dilation to fix our last interval. $[x, 2\pi - 2b) * 2^j$ must line up with our other interval, $[2n\pi - b, 2n\pi + x)$. So $2^j(2\pi - 2b) = 2n\pi - b$ and $2^{j+1}(x) = 2n\pi + x$ where $j \geq 2$. We get: $2^{j+1}\pi - 2n\pi = 2^{j+1}b - b$ and $2^{j+1}x - x = 2n\pi$.

    So, $b = \frac{(2^{j+1} - 2n)\pi}{2^{j+1} - 1}$ and $x = \frac{2n\pi}{2^{j+1} - 1}$.

    Thus we can write all 3 interval wavelet sets in the form of (12), and can say that any 3-interval wavelet set will be characterized only by a dilation factor $j \geq 2$ and a translation factor $n \geq 2$.

14. $y - 2^j\pi = x$ and $2^{j+1}x = y$. Thus re-write the wavelet set with only j as: $[-\frac{2^{2j+1}\pi}{2^{j+1}-1}, -2^j\pi) \cup [-\pi, -\frac{2^j\pi}{2^{j+1}-1}) \cup [\frac{2^j\pi}{2^{j+1}-1}, \pi) \cup [2^j\pi, \frac{2^{2j+1}\pi}{2^{j+1}-1})$.