Power Test
2004 Rice Math Tournament
February 28, 2004

Justify all answers, or give examples where appropriate. Partial credit will be given when appropriate.

Interval Notation

An interval from a to b of real numbers is denoted $[a, b)$ where this is the set $x : a = x < b$. A "[" means that the endpoint is included in the set while a "(" means that the endpoint is not included. So the interval $[\frac{\pi}{2}, \pi)$ is the set of real numbers between $\frac{\pi}{2}$ and π with $\frac{\pi}{2}$ included but π not included.

Partition:

Define a partition of the interval $[a, b)$ as a finite subset of points $x_0, x_1, ..., x_n$ such that $a = x_0$, $b = x_n$, and $x_i < x_{i+1}$ for all i such that $0 < i < n$. When we talk about a partition of a set into subsets, we mean the set of subsets of $[a, b) : [x_0, x_1), [x_1, x_2), ..., [x_{n-1}, x_n)$.

Translation congruent:

A set E is said to be 2π-translation congruent to $I_1 = [0, 2\pi)$ (denoted $E \sim_{2\pi} I_1$) if there is a partition of E into subsets such that adding a multiple of 2π to each subset will produce disjoint sets whose union is $[0, 2\pi)$.

Example: $[-\pi, -\frac{\pi}{2}) \cup [\frac{3\pi}{2}, 3\pi) \sim_{2\pi} [0, 2\pi)$ because the set $A = [-\pi, -\frac{\pi}{2}) + 2\pi \cup [\frac{3\pi}{2}, 2\pi) \cup [2\pi, 3\pi) - 2\pi = [\pi, \frac{3\pi}{2}) \cup [\frac{3\pi}{2}, 2\pi) \cup [0, \pi) = [0, 2\pi)$.

1. Show that $E = [-\frac{32\pi}{7}, -4\pi) \cup [-\pi, -\frac{4\pi}{7}) \cup [\frac{4\pi}{7}, \pi) \cup [4\pi, \frac{32\pi}{7})$ is 2π-translation congruent to $[0, 2\pi)$.

   Dilation congruent:

   A set E is said to be 2-dilation congruent to $I_2 = [-2\pi, -\pi) \cup [\pi, 2\pi)$ (denoted $E \sim_2 I_2$) if there is a partition of E into subsets such that multiplying each subset by a power of 2 will produce disjoint sets whose union is I_2.

   Example: $[-\frac{\pi}{2}, -\frac{\pi}{4}) \cup [\frac{7\pi}{4}, \frac{7\pi}{2})\ _2[-2\pi, -\pi) \cup [\pi, 2\pi)$ because the set $A = [-\frac{\pi}{2}, -\frac{\pi}{4}) * 2^2 \cup [\frac{7\pi}{4}, 2\pi) * 2^0 \cup [2\pi, \frac{7\pi}{2}) * 2^{-1} = [-2\pi, -\pi) \cup [\frac{7\pi}{4}, 2\pi) \cup [\pi, \frac{7\pi}{4}) = [-2\pi, -\pi) \cup [\pi, 2\pi)$.

2. Show that $E = [-\frac{32\pi}{7}, -4\pi) \cup [-\pi, -\frac{4\pi}{7}) \cup [\frac{4\pi}{7}, \pi) \cup [4\pi, \frac{32\pi}{7})$ is 2-dilation congruent to I_2.

   Wavelet Sets

   Define a wavelet set to be any set E which is 2π-translation congruent to I_1 and 2-dilation congruent to I_2. So the set from our first two problems is a wavelet set.

3. Show that the two example sets are also wavelets sets. (i.e. Show $[-\pi, -\frac{\pi}{2}) \cup [\frac{3\pi}{2}, 3\pi)\ 2[-2\pi, -\pi) \cup [\pi, 2\pi)$ and $[-\frac{\pi}{2}, -\frac{\pi}{4}) \cup [\frac{7\pi}{4}, \frac{7\pi}{2})\ _{2\pi}[0, 2\pi)$.)

   Prove:

4. If E is a wavelet set, then zero is not an element of E.

5. The interval $[a, b)$ is 2π-translation congruent to $[0, 2\pi)$ if and only if $b - a = 2\pi$.

6. For $a > 0$, the interval $[a, b)$ is 2-dilation congruent to $[\pi, 2\pi)$ if and only if $b = 2a$.

   1-interval wavelet sets

7. Prove that there can be no wavelet sets of just one interval (of the form $[a, b)$).

2-interval

Now study wavelet sets which are the union of 2 intervals. Find any wavelet sets which are of the form $E = [a, b) \cup [c, d)$ with $b < c$.

8. Prove $a = 2b$ and $d = 2c$ if the set is a wavelet set.
9. Write c as a function of b and using one variable write out the form that all 2 interval wavelet sets must have.

3-interval

Now we try to find all wavelet sets which are a union of 3 intervals. $[u, v) \cup [x, y) \cup [w, z)$

10. Reduce this case to finding 2 intervals $[x, y) \cup [w, z)$ which are 2π-translation congruent and 2-dilation congruent to one interval from the 2 interval case.
11. If one of the three intervals is $[-2b, -b)$ where $b > 0$, prove $b < \pi$.
12. Show that we need our sets to be of the form $[-2b, -b) \cup [x, 2\pi - 2b) \cup [2n\pi - b, 2n\pi + x)$ for some $n > 1$ and some $x > 0$.
13. Choose some dilation 2^j that makes the condition in (10) hold. Solve for x and b in terms of n and j.

4-interval

Define a 4-interval wavelet set of the form $[a, b) \cup [c, d) \cup [e, f) \cup [g, h)$ to be symmetric if $a = -h$, $b = -g$, $c = -f$, and $d = -e$. Now examine symmetric 4 interval wavelet sets of the form: $[-y, -2^j\pi) \cup [-\pi, -x) \cup [x, \pi) \cup [2^j\pi, y)$.

14. Write y in terms of x in two separate ways to solve for x and y as a function of j, and then write out the wavelet set in terms of only j.