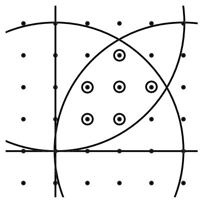
Algebra Test Solutions
2004 Rice Math Tournament
February 28, 2004

1. **Answer: 6**

   This is easiest to see by simply graphing the inequalities. They correspond to the (strict) interiors of circles of radius 4 and centers at $(0,0),(4,0),(0,4)$, respectively. So we can see that there are 6 lattice points in their intersection (circled in the figure).



2. **Answer: 6**

   For positive integers a, b, we have
   $$a! \mid b! \Leftrightarrow a! \le b! \Leftrightarrow a \le b.$$
   Thus,
   $$(n!)!)! \mid (2004!)! \Leftrightarrow (n!)! \le 2004! \Leftrightarrow n! \le 2004 \Leftrightarrow n \le 6.$$

3. **Answer: 8**

   Let $x = 2004$. Then the expression inside the floor brackets is
   $$\frac{(x+1)^3}{(x-1)x} - \frac{(x-1)^3}{x(x+1)} = \frac{(x+1)^4-(x-1)^4}{(x-1)x(x+1)} = \frac{8x^3+8x}{x^3-x} = 8 + \frac{16x}{x^3-x}.$$
   Since x is certainly large enough that $0 < 16x/(x^3-x) < 1$, the answer is 8.

4. **Answer: $\boldsymbol{x=10, y=5, z=4}$**

   Factoring, $(x-7)(y-3)=6, (x-7)(z-2)=6, (y-3)(z-2)=4$. This implies that
   $$\begin{aligned} x-7 &= 3 \\ y-3 &= 2 \\ z-2 &= 2 \end{aligned}$$
   Thus $x=10, y=5, z=4$.

5. **Answer: $\frac{-1+\sqrt{5}}{2}$**

   Draw a right triangle with legs $1, x$; then the angle θ opposite x is $\tan^{-1} x$, and we can compute $\cos(\theta) = \frac{1}{\sqrt{x^2+1}}$. Thus, we only need to solve $x = \frac{1}{\sqrt{x^2+1}}$. This is equivalent to $x\sqrt{x^2+1} = 1$. Square both sides to get $x^4 + x^2 = 1 \Rightarrow x^4 + x^2 - 1 = 0$. Use the quadratic formula to get the solution $x^2 = \frac{-1+\sqrt{5}}{2}$ (unique since x^2 must be positive).

6. **Answer: 2hours.**

   Adding the individual rates, we get $\frac{1}{3} + \frac{1}{10} + \frac{1}{15} = \frac{1}{2}$ of the room is cleaned per hour so the whole room takes two hours.

7. **Answer: $\mathbf{0 < x < 1}$ or $\mathbf{2 < x < 3}$ or $\mathbf{4 < x < 5}$**

   The sign of one of the terms switches every time x moves from the range $(I, I+1)$ to $(I+1, I+2)$. When x is less than zero, all terms are negative so the LHS is negative. Also note that x is undefined at $1, 3, 5$.

8. **Answer: 128**

   For any integer $n \geq 0$, the given implies $x^{n+3} = -4x^{n+1} + 8x^n$, so we can rewrite any such power of x in terms of lower powers. Carrying out this process iteratively gives

$$\begin{aligned} x^7 &= -4x^5 + 8x^4 \\ &= 8x^4 + 16x^3 - 32x^2 \\ &= 16x^3 - 64x^2 + 64x \\ &= -64x^2 + 128. \end{aligned}$$

   Thus, our answer is 128.

9. **Answer: 677**

   If d is the relevant greatest common divisor, then $a_{1000} = a_{999}^2 + 1 \equiv 1 = a_0 \pmod{d}$, which implies (by induction) that the sequence is periodic modulo d, with period 1000. In particular, $a_4 \equiv a_{2004} \equiv 0$. So d must divide a_4. Conversely, we can see that $a_5 = a_4^2 + 1 \equiv 1 = a_0$ modulo a_4, so (again by induction) the sequence is periodic modulo a_4 with period 5, and hence a_{999}, a_{2004} are indeed both divisible by a_4. So the answer is a_4, which we can compute directly; it is 677.

10. **Answer: $-\frac{\mathbf{2010012}}{\mathbf{2010013}}$**

   Let $z_1, \ldots, z_5$ be the roots of $Q(z) = z^5 + 2004z - 1$. We can check these are distinct (by using the fact that there's one in a small neighborhood of each root of $z^5 + 2004z$, or by noting that $Q(z)$ is relatively prime to its derivative). And certainly none of the roots of Q is the negative of another, since $z^5 + 2004z = 1$ implies $(-z)^5 + 2004(-z) = -1$, so their squares are distinct as well. Then, $z_1^2, \ldots, z_5^2$ are the roots of P, so if we write C for the leading coefficient of P, we have

$$\begin{aligned} \frac{P(1)}{P(-1)} &= \frac{C(1-z_1^2)\ldots(1-z_5^2)}{C(-1-z_1^2)\ldots(-1-z_5^2)} \\ &= \frac{[(1-z_1)\ldots(1-z_5)]\cdot[(1+z_1)\ldots(1+z_5)]}{[(i-z_1)\ldots(i-z_5)]\cdot[(i+z_1)\ldots(i+z_5)]} \\ &= \frac{[(1-z_1)\ldots(1-z_5)]\cdot[(-1-z_1)\ldots(-1-z_5)]}{[(i-z_1)\ldots(i-z_5)]\cdot[(-i-z_1)\ldots(-i-z_5)]} \\ &= \frac{(1^5+2004\cdot 1-1)(-1^5+2004\cdot(-1)-1)}{(i^5+2004\cdot i-1)(-i^5+2004\cdot(-i)-1)} \\ &= \frac{(2004)(-2006)}{(-1+2005i)(-1-2005i)} \\ &= -\frac{2005^2-1}{2005^2+1} \\ &= -\frac{4020024}{4020026} = -\frac{2010012}{2010013}. \end{aligned}$$