Team Solutions
2003 Rice Math Tournament
February 22, 2003

1. **Answer: 2 or 2 : 1 or 2 to 1**

   Let ABC be an equilateral triangle of side length s. Place triangle ABC on a coordinate grid so that $A = (-\frac{s}{2}, 0), B = (\frac{s}{2}, 0)$, and $C = (0, \frac{s\sqrt{3}}{2})$. Clearly one edge of the rectangle must lie on an edge of the triangle, so assume this is AB. Label the vertex of the rectangle on BC (x, y). Note that the equation of BC is $y = -\sqrt{3}x + \frac{s\sqrt{3}}{2}$. The area of the rectangle is $2xy = -2\sqrt{3}x^2 + s\sqrt{3}x = -2\sqrt{3}(x - \frac{s}{4})^2 + 2\sqrt{3}(\frac{s}{4})^2$. Maximum area occurs when $x = \frac{s}{4}$ and the area is then $\frac{s^2\sqrt{3}}{8}$. The area of the triangle is $\frac{s^2\sqrt{3}}{4}$, and thus the ratio is 2.

2. **Answer: 25, 599, 869, 679**

   The near symmetry of the polynomial suggests we look at binomial coefficients. Using Pascal's triangle, we can quickly note that $P(x) = (x+1)^{12} - 160000x^4$.

3. **Answer: $(R, B, Y, G), (B, Y, R, G), (G, R, B, Y), (Y, G, B, R)$**

   There are many approaches but perhaps the fastest is a graph theoretical approach. Draw a graph on 4 vertices where the edges are labeled. An edge is drawn between two vertices if they appear on opposite sides of a cube. The edge is labeled with the number of the cube it comes from. Then what we are looking for is two subgraphs that are each regular of degree 2 with four edges, one of each number. The two subgraphs must be disjoint. One subgraphs tells us which pairs of colors align on the left and right side. The other would give us the front and back. We can easily find such a solution and in this case it is unique. Starting at an arbitrary side and going clockwise the solution is (R, B, Y, G), (B, Y, R, G), (G, R, B, Y), (Y, G, B, R).

4. **Answer: 499**

   The given sum is equivalent to $\sum_{k=1}^{2002} [(k+1)! - k!]$ which is equal to $2003! - 1!$. Now, clearly 2003! will end with a lot of zeros and when we subtract the 1 we get a series of 9s. We need to see how many factors of 10 divide 2003!. We have many more 2s than 5s, so 5s are the limiting factor. There are $\lfloor \frac{2003}{5} \rfloor + \lfloor \frac{2003}{5^2} \rfloor + \lfloor \frac{2003}{5^3} \rfloor + \lfloor \frac{2003}{5^4} \rfloor = 400 + 80 + 16 + 3 = 499$ 5s we can take out of 2003!, so 499 factors of 10 divide 2003!, so the number ends with 499 9s.

5. **Answer: 447**

   Let $a_n = \frac{200003^n}{(n!)^2}$. If a_n is the maximum, then $a_{n-1} \le a_n$ and $a_n \ge a_{n+1}$. The first inequality implies $\frac{200003^{n-1}}{((n-1)!)^2} \le \frac{200003^n}{(n!)^2}$ which is equivalent to $n^2 \le 200003$. The second inequality implies similarly that $(n+1)^2 \ge 200003$. The only n for which these two inequalities are true is $n = 447$.

6. **Answer: 25499475**

   $x^3 = (x-1)^3 + (x-1)^2 + (x-1)(x) + x^2$. Thus, we can write $x^3 + (x-1)^3 = (x-1)^2 + (x-1)(x) + x^2 + 2(x-1)^3 = x^2 + x(x-1) + 3(x-1)^2 + 2(x-1)(x-2) + 2(x-2)^2 + 2(x-2)^3$. Continuing this process we can generalize to $\sum_{i=1}^{x} i^3 = (x^2 + 3(x-1)^2 + 5(x-2)^2 + 7(x-3)^2 + \cdots) + (x(x-1) + 2(x-1)(x-2) + 3(x-2)(x-3) + \cdots)$. The righthand side can be written in sum notation, gathering together by powers of x, as $(x^2 \sum_{i=1}^{x}(2i-1) - 2x \sum_{i=1}^{x}(i-1)(2i-1) + \sum_{i=1}^{x}(2i-1)(i-1)^2) + (x^2 \sum_{i=1}^{x-1} i - x \sum_{i=1}^{x-1}(i)(i-1+i) + \sum_{i=1}^{x-1}(i)(i-1)(i))$. Thus $\sum_{i=1}^{x} i^3 = -\frac{1}{2}x^2(x^2 + 2x + 1) + 3\sum_{i=1}^{x} i^3$. This can be simplified to $\sum_{i=1}^{x} i^3 = \frac{x^2(x+1)^2}{4}$. Thus $11^3 + 12^3 + \cdots + 100^3 = s(100) - s(10) = 25502500 - 3025 = 25499475$.

7. **Answer: $\frac{1111}{7776} = \frac{1111}{6^5}$**

   Consider the more general problem of rolling n dice. Let P_n be the probability that the sum is divisible by 7. Suppose we know the sum of $n-1$ of the n dice. If this sum is divisible by 7, then the sum of all n cannot be. If this sum is not divisible by 7, then there is exactly one number on the n^{th} die that will work. Thus, P_n = probability the first $n-1$ are not divisible by 7 times the probability of rolling what we need on the n^{th} die. So $P_n = (1 - P_{n-1})\frac{1}{6}$. Since $P_1 = 0$ and $P_2 = \frac{1}{6}$, we get $P_3 = \frac{5}{36}, P_4 = \frac{31}{216}, P_5 = \frac{185}{1296}$ and $P_6 = \frac{1111}{7776}$.

8. **Answer: $\frac{105}{512}$**

   Together the students answered at least $2+1+5+4+1+0+3=16$ questions correctly. In addition, all students have a number of questions of which they are are completely uncertain whether they answered them correctly (Christi has 0, Barbara and David have 1, Allison, Ed, Fred, and Gary have 2), so that each student assigns a 50% probability to the event that he or she answered correctly. There are 10 such questions. The probability that the students' scores totalled 20 is the probability that exactly 4 of these 10 questions were answered correctly, which equals $\binom{10}{4} \cdot \left(\frac{1}{2}\right)^{10} = \frac{210}{1024} = \frac{105}{512}$.

9. **Answer: 129**

   Consider a $3 \times n$ chessboard. A corner square can be covered in two ways, and a moment's thought reveals that covering the corner with a domino parallel to the "3" side yields F_{n-1} arrangements, whereas covering the corner with a domino parallel to the "n" side yields F_{n-3} arrangements. We therefore have the recurrence relation $F_n = F_{n-1} + F_{n-3}$ and the initial conditions $F_1 = F_2 = 1$, $F_3 = 2$. F_{14} is easily computed to be 129.

10. **Answer: 2**

    The possible moves are $\{1, 2, 3, 5, 7, 11\}$. If Kate takes 11 sticks, then she has taken the last stick and lost. If she takes 7 sticks, then Adam can take 3 sticks leaving Kate to take the last remaining stick. If Kate takes 5 or 3 sticks, then Adam will take 5 or 7 sticks, respectively, leaving Kate the last stick. Thus Kate's only possible winning starting choices are 1 or 2. If the first move is 1, then Adam should take 5 sticks, leaving 5. Now, if Kate takes 1, 2, or 3 sticks, Adam can take 3, 2, or 1 respectively, always leaving Kate the last stick. Thus Kate should always take 2 sticks initially. We should show this is a win. If Adam takes 7, 5, 3 or 1 sticks after Kate takes 2 sticks, then Kate can take 1, 3, 5, or 7 respectively, leaving Adam one stick. If Adam takes 2 sticks after Kate took 2 sticks, then Kate should take 2 sticks again leaving 5 sticks. Then Adam must take 1, 2 or 3 sticks. Kate should take 3, 2 or 1 stick respectively always leaving one last stick for Adam. Thus, the only way for Kate to win is to take 2 sticks initially, so the answer is 2.

11. **Answer: $\left(\frac{\sqrt{2}}{4}(1+\sqrt{3}), \frac{\sqrt{2}}{4}(-1+\sqrt{3})\right), \left(\frac{\sqrt{2}}{4}(1-\sqrt{3}), \frac{\sqrt{2}}{4}(1+\sqrt{3})\right), \left(\frac{\sqrt{2}}{4}(-1-\sqrt{3}), \frac{\sqrt{2}}{4}(1-\sqrt{3})\right)$, and $\left(\frac{\sqrt{2}}{4}(-1+\sqrt{3}), \frac{\sqrt{2}}{4}(-1-\sqrt{3})\right)$**

    First, we need to note that if we set $z = x+iy$ where $i = \sqrt{-1}$ then $z^2 = f(x,y)+ig(x,y)$. Therefore, we want to solve for z such that $z^4 = f^2 - g^2 + 2ifg = \frac{1}{2} + i\frac{\sqrt{3}}{2}$. Note that the complex number on the right can be drawn in the complex plane as a vector with magnitude 1 and angle $\frac{\pi}{3}$ with the x-axis. Thus $z^4 = e^{i\frac{\pi}{3}} = \cos\left(\frac{\pi}{3} + 2\pi k\right) + i\sin\left(\frac{\pi}{3} + 2\pi k\right)$. We can see then that $z = \cos\left(\frac{\pi}{12} + \frac{\pi k}{2}\right) + i\sin\left(\frac{\pi}{12} + \frac{\pi k}{2}\right)$, where $k = 0, 1, 2$ or 3. We can solve for these sine and cosine values exactly using that $\sin\frac{\pi}{12} = \sin\left(\frac{\pi}{3} - \frac{\pi}{4}\right) = \frac{\sqrt{2}}{4}(\sqrt{3}-1)$ and $\cos\frac{\pi}{12} = \frac{\sqrt{2}}{4}(1+\sqrt{3})$. All four points are $\left(\frac{\sqrt{2}}{4}(1+\sqrt{3}), \frac{\sqrt{2}}{4}(-1+\sqrt{3})\right), \left(\frac{\sqrt{2}}{4}(1-\sqrt{3}), \frac{\sqrt{2}}{4}(1+\sqrt{3})\right), \left(\frac{\sqrt{2}}{4}(-1-\sqrt{3}), \frac{\sqrt{2}}{4}(1-\sqrt{3})\right)$, and $\left(\frac{\sqrt{2}}{4}(-1+\sqrt{3}), \frac{\sqrt{2}}{4}(-1-\sqrt{3})\right)$.

12. **Answer: 16**

    First, since we are summing the segment changes in a complete cycle, the total number of changes must be even. Now, we can define a distance function between any two digits which is the number of segment changes to change from one to the other. Considering that many digits are at least 2 changes from any other digit or a distance of 1 from only one digit, one can easily prove that the absolute minimum number of segment changes is 16. That 16 is actually achieved is easy. The ordering 3, 7, 1, 4, 9, 5, 6, 0, 8, 2 has 16 total changes.

13. **Answer: 18**

    Rewriting $(\cos 10x)(\cos 9x)$ as $\frac{1}{2}(\cos x + \cos 19x)$ we can visualize the graph as a cosine wave $(\cos 19x)$ with period $\frac{2\pi}{19}$ oscillating within an envelope set by $\frac{1}{2}(\cos x - 1) \leq y \leq \frac{1}{2}(\cos x + 1)$. Clearly once the top of the envelope is below $\frac{1}{2}$ $(\cos x < 0)$, then there are no solutions. Thus, we only need to consider the intervals $[0, \frac{\pi}{2}]$ and $[\frac{3\pi}{2}, 2\pi]$. The function is even around $x = \pi$ so we can just consider the first interval and double our answer. In this interval $\cos 19x$ goes through 9.5 half-cycles, each crossing $\frac{1}{2}$. This gives us 9 solutions (the last half of a half-cycle does not reach $\frac{1}{2}$) in the first interval for a total of 18 solutions.

14. **Answer: $\mathbf{2^{2001} - 2^{1000}}$**

Using the binomial theorem, we can see $2^{2003} = (1+1)^{2003} = \binom{2003}{0} + \binom{2003}{1} + \binom{2003}{2} + \binom{2003}{3} + \binom{2003}{4} + \cdots$. Note also that $(1+i)^{2003} = \binom{2003}{0} + i\binom{2003}{1} - \binom{2003}{2} - i\binom{2003}{3} + \binom{2003}{4} + \cdots$. We can also use the equations $0 = (1-1)^{2003} = \binom{2003}{0} - \binom{2003}{1} + \binom{2003}{2} - \binom{2003}{3} + \binom{2003}{4} + \cdots$ and $(1-i)^{2003} = \binom{2003}{0} - i\binom{2003}{1} - \binom{2003}{2} + i\binom{2003}{3} + \binom{2003}{4} + \cdots$. Adding these four equations yields $2^{2003} + (1+i)^{2003} + (1-i)^{2003} = 4(\binom{2003}{0} + \binom{2003}{4} + \binom{2003}{8} + \cdots) = 4S$ if we call the sum we desire S. Note that $(1+i)$ is a complex number with a magnitude of $\sqrt{2}$ and an angle of $\frac{\pi}{4}$. Therefore, $(1+i)^{2003} = (\sqrt{2}e^{i\frac{\pi}{4}})^{2003} = 2^{\frac{2003}{2}}e^{\frac{i2003\pi}{4}}$ which has magnitude $2^{\frac{2003}{2}}$ and angle $\frac{2003\pi}{4} = \frac{3\pi}{4}$. Thus $(1+i)^{2003} = 2^{\frac{2003}{2}}(\frac{-1+i}{\sqrt{2}}) = 2^{1001}(-1+i)$. Similarly, $(1-i)^{2003} = 2^{1001}(-1-i)$. Thus $4S = 2^{2003} + 2^{1001}(-1+i-1-i) = 2^{2003} - 2^{1002}$. Solving for S yields $S = 2^{2001} - 2^{1000}$.

15. **Answer: $\mathbf{\frac{916709}{2^4 \cdot 3^4 \cdot 5^3 \cdot 7^2 \cdot 19}}$**

If the player had no cards, then clearly the probability of guessing correctly is $\frac{1}{9} \cdot \frac{1}{6} \cdot \frac{1}{6}$. However, each of the four cards in his hand cannot be one of the hidden cards. The difficulty lies in that the distribution of his hand (the number of people cards versus the number of weapon cards versus the number of room cards) affects the probability of guessing correctly. Call the probability of getting a given distribution P_d where d is a triplet of numbers (a, b, c) representing each possible distribution. Note $a + b + c = 4$. Call the probability, given distribution d, of guessing the three hidden cards G_d. Then the answer is $\Sigma P_d G_d$ where the sum runs over all the different distributions. The rest is calculations.