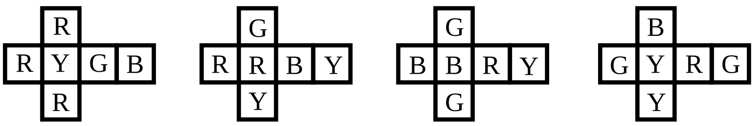
Team Test
2003 Rice Math Tournament
February 22, 2003

1. What is the ratio of the area of an equilateral triangle to the area of the largest rectangle that can be inscribed inside the triangle?

2. Define $P(x) = x^{12} + 12x^{11} + 66x^{10} + 220x^9 + 495x^8 + 792x^7 + 924x^6 + 792x^5 - 159505x^4 + 220x^3 + 66x^2 + 12x + 1$. Find $\frac{P(19)}{20^4}$.

3. Four flattened colored cubes are shown below. Each of the cubes' faces has been colored red (R), blue (B), green (G) or yellow (Y). The cubes are stacked on top of each other in numerical order with cube #1 on bottom. The goal of the puzzle is to find an orientation for each cube so that on each of the four visible sides of the stack all four colors appear. Find a solution, and for each side of the stack, list the colors from bottom to top. List the sides in clockwise order.



Cube 1 Cube 2 Cube 3 Cube 4

4. When evaluated, the sum $\sum_{k=1}^{2002} [k \cdot k!]$ is a number that ends with a long series of 9s. How many 9s are at the end of the number?

5. Find the positive integer n that maximizes the expression $\frac{200003^n}{(n!)^2}$.

6. Find $11^3 + 12^3 + ... + 100^3$. Hint: Develop a formula for $s(x) = \sum_{i=1}^{x} i^3$ using perhaps $x^3 - (x-1)^3$. Also the formulas $\sum_{i=1}^{x} i = \frac{x(x+1)}{2}$ and $\sum_{i=1}^{x} i^2 = \frac{x(x+1)(2x+1)}{6}$ may be helpful.

7. Six fair 6-sided dice are rolled. What is the probability that the sum of the values on the top faces of the dice is divisible by 7?

8. Several students take a quiz which has five questions, and each one is worth a point. They are unsure as to how many points they received, but all of them have a reasonable idea about their scores. Below is a table of what each person thinks is the probability that he or she got each score. Assuming their probabilities are correct, what is the probability that the sum of their scores is exactly 20?

Student \ Score	0	1	2	3	4	5
Allison	0	0	.25	.5	.25	0
Barbara	0	.5	.5	0	0	0
Christi	0	0	0	0	0	1
David	0	0	0	0	.5	.5
Ed	0	.25	.5	.25	0	0
Fred	.25	.5	.25	0	0	0
Gary	0	0	0	.25	.5	.25

9. Let F_n be the number of ways of completely covering an $3 \times n$ chessboard with n 3×1 dominoes. For example, there are two ways of tiling a 3×3 chessboard with three 3×1 dominoes (all horizontal or all vertical). What is F_{14}?

10. Two players (Kate and Adam) are playing a variant of Nim. There are 11 sticks in front of the players and they take turns each removing either one or any prime number of sticks. The player who is forced to take the last stick loses. The problem with the game is that if player one (Kate) plays perfectly, she will always win. Give the sum of all the starting moves that lead to a sure win for Kate (assuming each player plays perfectly).

11. Define $f(x,y) = x^2 - y^2$ and $g(x,y) = 2xy$. Find all (x,y) such that $(f(x,y))^2 - (g(x,y))^2 = \frac{1}{2}$ and $f(x,y) \cdot g(x,y) = \frac{\sqrt{3}}{4}$.

12. The numerals on digital clocks are made up of seven line segments. When various combinations of them light up different numbers are shown. When a digit on the clock changes, some segments turn on and others turn off. For example, when a 4 changes into a 5 two segments turn on and one segment turns off, for a total of 3 changes. In the usual ordering $1, 2, 3, \ldots, 0$ there are a sum total of 32 segment changes (including the wrapping around from 0 back to 1). If we can put the digits in any order, what is the fewest total segment changes possible? (As above, include the change from the last digit back to the first. Note that a 1 uses the two vertical segments on the right side, a 6 includes the top segment, and a 9 includes the bottom segment.)

13. How many solutions are there to $(\cos 10x)(\cos 9x) = \frac{1}{2}$ for $x \in [0, 2\pi]$?

14. Find $\binom{2003}{0} + \binom{2003}{4} + \binom{2003}{8} + \cdots$. Hint: Consider $(1 \pm i)$ and (1 ± 1).

15. Clue is a board game in which four players attempt to solve a mystery. Mr. Boddy has been killed in one of the nine rooms in his house by one of six people with one of six specific weapons. The board game has 21 cards, each with a person, weapon or room on it. At the beginning of each game, one room card, one person card and one weapon card are randomly chosen and set aside as the solution to the mystery. The remaining cards are all gathered, shuffled, and dealt out to the four players. Note that two players get four cards and two get five cards. Players thus start out knowing only that the cards in their hand are not among the solution cards. Essentially, players then take turns guessing the solution to the mystery. A guess is a listing of a person, weapon and a room. Assume that the player who goes first in the game has only 4 cards. What is the probability that his first guess, the first guess of the game, is exactly right (i.e. he guesses all three hidden cards correctly)?