Power Test
2003 Rice Math Tournament
February 22, 2003

Justify all answers, or give examples where appropriate. Partial credit will be given when appropriate. In this power test, it is not necessary to simplify $\binom{n}{k}$ or x^y in your answers, although you will not be penalized for doing so. For example, $\binom{40}{10} \cdot 3^{50}$ is an acceptable form for an answer.

Combinatorial Arguments

$\binom{n}{k}$ is the number of possible combinations of n objects taken k at a time (i.e. the number of subsets of $\{1, 2, \dots, n\}$ that have k elements). Note that in a combination, the order of elements does not matter. Unless otherwise specified, combinations are not allowed to have the same element more than once. $\binom{n}{k} = \frac{n!}{(n-k)! \cdot k!}$, where $n!$ (n factorial) is $n \cdot (n-1) \cdot (n-2) \cdot \dots \cdot 2 \cdot 1$. Note that $1!$ and $0!$ are defined to be 1.

A **combinatorial argument** is a way of proving two expressions are equal by showing they are two different ways of counting the same thing.

1. Simplify
$$\sum_{n=0}^{100} \binom{100}{n} \cdot 2^n$$
and show why your answer is correct. (Hint: the answer can be written in the form x^y, for some integers x, y)

2. Find the number of 30-digit combinations (where repetition is allowed) from the set $\{0, 1, 2, 3, 4, 5\}$.

3. Find the number of 30-digit combinations (where repetition is allowed) from the set $\{0, 1, 2, 3, 4, 5\}$ in which each digit i in the set occurs at least i times in the combination.

4. Using a combinatorial argument, find
$$\binom{k}{k} + \binom{k+1}{k} + \dots + \binom{n}{k}$$
(where $k \le n$) in terms of n and k, and show why your answer is correct. (Hint: the answer can be written in the form $\binom{x}{y}$ for some x, y)

5. Show that
$$\binom{n}{m}\binom{m}{k} = \binom{n}{k}\binom{n-k}{m-k}$$
for $k \le m \le n$ using a combinatorial argument.

6. Simplify $\displaystyle\sum_{i=0}^{20} \binom{50}{i}\binom{50-i}{20-i}$.

7. Find $\binom{100}{1} + 2 \cdot \binom{100}{2} + 3 \cdot \binom{100}{3} + \dots + 100 \cdot \binom{100}{100}$, and show why your answer is correct.

Stirling Numbers

Define the *Stirling numbers of the second kind* $S(m, n)$ as the number of ways to break an m-element set into n nonempty parts (note that the order of the parts does not matter). For example, $S(3, 2) = 3$, since we can break $\{1, 2, 3\}$ into 2 nonempty parts in the following ways: $\{1\}\{2, 3\}$, $\{2\}\{1, 3\}$, $\{3\}\{1, 2\}$, which gives a total of 3 ways to do this.

8. Find the following values of the Stirling numbers of the second kind (no proof is needed):
   (a) $S(4, 3)$
   (b) $S(5, 2)$
   (c) $S(6, 2)$

9. Find a general formula for $S(m, 2)$ (where $m > 1$) and prove that it is correct.
10. Find a general formula for $S(m, m-1)$ (where $m > 1$) and prove that it is correct.
11. Find a recursive formula for $S(m, n)$ and prove that it is correct.

Card Probability

A group of college students are playing a game which uses one or more regular decks of playing cards in the following way. The game requires one suit (thirteen cards, call them one through thirteen) per player. For example, six players would play with one full deck (4 suits) and two more suits from another deck. Threes are the highest ranked cards in the game. Each round all the cards are shuffled and dealt out to the players (so each player gets exactly thirteen cards).

12. If there are two players, Alice and Bob, what is the probability that Alice will be dealt no threes in any given round?
13. If there are two players, Alice and Bob, what is the probability that Alice or Bob will be dealt no threes in a given round?
14. If there are three players, Alice, Bob, and Carol, what is the probability that either Alice, Bob, or Carol will be dealt no threes in a given hand? (It is allowed for more than one of them to get no threes).
15. N players are playing and Alice seems to be having a run of bad luck. Every hand she has no threes. If she wants to improve her chances of getting at least one three, does she want more or fewer players in the game? Justify your answer.
16. As more and more people join in the game, what happens to the probability that Alice is dealt no threes in a given hand?

Designs

In mathematics, a design is a precise combinatorial object defined as the set of two objects P and B. P is a set of v points, and B is a set of sets called blocks. Every block has the same size (k) and is a subset of k points from P. Lastly, a design must have this property: For every pair of distinct points in P, there are exactly λ blocks that contain both points. Note that what the points of P are called is unimportant, only the number of points, so assume P is the numbers 1 to v. Typically a design is labeled by the parameters (v, k, λ), but multiple designs can exist with the same parameters, so the actual blocks of the design are significant.

An obvious use of designs is taste tests. Consider P to be a set of v different beverages. You probably do not want each person to test all the drinks, especially if v is large. Instead, have each person test the drinks contained in one of the blocks. Thus, each person tries k drinks. The last requirement of a design makes sure that every two drinks are compared the same number of times. That each pair of drinks should be compared is clear; requiring the same number of comparison for each pair gives a very symmetric object.

Here is an example of a (3,2,1) design. Let $P = \{1, 2, 3\}$ and $B = \{B_1, B_2, B_3\}$, where $B_1 = \{1, 2\}, B_2 = \{2, 3\}$, and $B_3 = \{1, 3\}$. Note that there are three blocks and three points in P; this is just a coincidence.

17. Give an example of a (7,3,1) design.
18. Let b be the number of blocks in a given (v, k, λ) design. Prove that $b = \frac{\lambda\binom{v}{2}}{\binom{k}{2}}$. Note that this surprisingly shows that every (v, k, λ) design for a given v, k and λ has the same number of blocks.
19. Let r be the number of blocks any given point in P is in. Note that r does not depend on the particular point nor on the particular design, just on v, k and λ. Determine and prove r's relationship with v, k and λ.
20. A design is trivial if it has one block containing all the points of P, or if it has all subsets of P of size k as blocks. Show that in any nontrivial $(v, k, 1)$ design, $v \geq 3(k-1)$.