Calculus Test
2003 Rice Math Tournament
February 22, 2003

1. Given $f(x) = \frac{1}{x}$, find $f^{(100)}(x)$.

2. A windup penguin moves along the x-axis with acceleration given by $a(t) = 2t - 2$ units per second. At $t = 1$ second, the penguin is moving left with a speed of 4 units per second. What is the total distance the penguin travels in the first four seconds ($t = 0$ to $t = 4$)?

3. The function

$$f(x) = \begin{cases} 2 + x^3, x \leq 1 \\ 3x, x > 1 \end{cases}$$

is differentiable on $(-1, 2)$ and continuous on $[-1, 2]$, and hence satisfies the hypotheses for the Mean Value Theorem on the interval $[-1, 2]$. Find a value of x which is guaranteed to exist by the theorem.

4. Let $f(x) = (x-1)(x-2)(x-3)^2(x-4)(x-5)(x-6)$. Find $f''(3) - f'(3) + f(3)$.

5. Consider a function $f : \mathbf{R} \to \mathbf{R}$. Given that $f(0) = 0$, $\lim_{h \to 0} \dfrac{f(h)}{h} = 7$, and $f(x + y) = f(x) + f(y) + 3xy$ for all $x, y \in \mathbf{R}$, what is $f(7)$?

6. A group of college students gets together to write math questions while eating cake. Teena eats $c(t)$ pieces of cake per hour and writes $q(t)$ questions per hour. The faster Teena eats cake, the less she wants to eat, so $c'(t) = -c(t)$. However, as she eats, she gets sleepy and writes questions more slowly, so $q'(t) = -3c(t)$.

   When the question writing session begins (at $t = 0$), Teena is writing 7 questions per hour and eating 2 pieces of cake per hour. But if she ever writes questions more slowly than 2 per hour, she gets kicked out. The question writing session lasts for 24 hours and thus ends at $t = 24$. Does Teena get kicked out of the question writing session before it ends? And if so, when?

7. Find the derivative of x^{x^x}.

8. Let $s > 0$. Consider straight lines drawn from $(0, r)$ to $(s - r, 0)$ for all $0 \leq r \leq s$. We define $f(x)$ as the maximum y-value that any of these lines take on at x. Find $f(x)$ for $0 \leq x \leq s$.

9. An hourglass (a shape composed of two cones of the same size joined at the apex oriented so that the hourglass sits on one of the cones' base) has a maximal radius of 12 inches and a total height of 10 inches. The sand inside occupies one-fourth of the total volume of the hourglass, and it takes one minute for all the sand to drop from the top cone to the bottom one (the hourglass is a one minute timer). Assuming the sand falls at a constant rate, what is the rate of change in the height of the sand in the bottom cone 15 seconds before the timer runs out? Assume the surface of the sand in the bottom cone is always flat.

10. Sammy the Owl is making a one-eyed Jack'o'Lantern. He bought a perfectly spherical pumpkin of radius 1 foot. He first carves out the inside so that the inner radius is 10 inches. He then drills a perfectly circular hole of radius 1 inch for the eye. He drills straight toward the center of the pumpkin at the "equator." What is the total volume in cubic inches of pumpkin that he has removed?