

ALGEBRA TEST
2003 RICE MATH TOURNAMENT
FEBRUARY 22, 2003

1. Let a and b be real numbers, and suppose that $a * b = a^2 + ab + 3b + 1$. List all numbers a such that there is no b for which $a * b = 2$.
2. What is the smallest positive number k such that there are real numbers a and b satisfying $a + b = k$ and $ab = k$?
3. Assume the polynomial $p(x) = x^8 - 12x^6 + 49x^4 - 78x^2 + 42$ has no complex roots. How many negative real roots does it have?
4. Harry, Hermione, and Ron go to Diagon Alley to buy chocolate frogs. If Harry and Hermione spent one-fourth of their own money, they would spend 3 galleons in total. If Harry and Ron spent one-fifth of their own money, they would spend 2 galleons in total. Everyone has a whole number of galleons, and the number of galleons between the three of them is a multiple of 7. What are all the possible numbers of galleons that Harry can have?
5. In the following equation, x and y are digits and the subscripts are number bases. $(11xy)_7 = (310x)_5$. Find (x, y) .
6. Patty is picking peppermints off a tree. They come in two colors, red and white. She picks fewer than 100 total peppermints but at least one of each color. The white flavor is stronger, so she prefers red to white. Thus, she always picks fewer white peppermints than ten times the number of reds. How many different combinations of peppermints can she go home with?
7. Let r_1, r_2 , and r_3 be the solutions of the equation $x^3 - 2x^2 + 4x + 10 = 0$. Compute $(r_1 + 2)(r_2 + 2)(r_3 + 2)$.
8. Solve for x :

$$\log_2 \log_4 x + \log_4 \log_2 x = 2.$$

9. Find all real values of x which satisfy $\frac{4}{|x|+1} \geq 1 + |2|x| - 4|$.
10. Let $b > 0$ and $c > 0$. Suppose that the sequence x_1, x_2, x_3, \dots is defined by

$$\begin{aligned}x_0 &= 1 \\x_1 &= 1 \\x_{n+2} &= bx_{n+1} + cx_n, \quad n \geq 0.\end{aligned}$$

The ratio x_{n+1}/x_n approaches a finite number R as n goes to ∞ . What is R ?