Advanced Topics Solutions
2003 Rice Math Tournament
February 22, 2003

1. **Answer:** $\mathbf{\frac{23}{87}}$

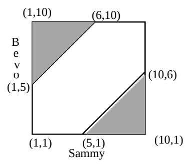
   Take $y = 1$. Then $(x \cdot 1^2)\diamondsuit 1 = x(1\diamondsuit 1) = x$ by the first rule. Thus $x\diamondsuit 1 = x$. Now $x\diamondsuit y = (\frac{x}{y^2} \cdot y^2)\diamondsuit y = \frac{x}{y^2}(y\diamondsuit 1) = \frac{x}{y^2} \cdot y = \frac{x}{y}$. Thus $23\diamondsuit 87 = \frac{23}{87}$.

2. **Answer:** $\mathbf{\frac{1}{1025}}$

   The probability that the coin is fair is $\frac{A}{A+B}$ where A is the number of ways to obtain 10 consecutive heads with the fair coin and B is the number of ways with the unfair coin. $A = 1$ and $B = 2^{10}$. Thus the answer is $\frac{1}{1025}$.

3. **Answer:** $\mathbf{\frac{25}{81}}$

   Call Sammy's number s and Bevo's number b. They differ by more than 4 if $b - s > 4$ or $s - b > 4$. All possible choices can be represented by the square shown below. The shaded region is where they differ by more than 4 and it has area $\frac{25}{2} + \frac{25}{2} = 25$. The probability is then $\frac{25}{9^2} = \frac{25}{81}$.



4. **Answer:** $\mathbf{4 + 2\log_2 3}$ **or** $\mathbf{4 + \log_2 9}$ **or** $\mathbf{\log_2 144}$

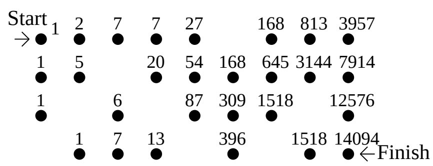
   From their roots, we know that $b(x) = c_1(x+7)(x+11)$, $c(x) = c_2(x+5)(x+8)$, and $d(x) = c_3(x+5)(x+11)$ for some c_1, c_2, c_3. Since $\log_2 a(x) = \log_2 \frac{b(x)c(x)}{d(x)} = \log_2 \frac{c_1c_2(x+7)(x+11)(x+5)(x+8)}{c_3(x+11)(x+5)}$, then we see that $\log_2 a(x) = \log_2(c_4(x+7)(x+8))$, where $c_4 = \frac{c_1c_2}{c_3}$. So, $a(x) = c_4(x+7)(x+8)$. Since $a(-6) = 4$, then we see that $c_4 = 2$, so $a(x) = 2(x+7)(x+8)$, and $a(1) = 2 \cdot 8 \cdot 9 = 144$. So, $\log_2 a(1) = \log_2 144 = 4 + \log_2 9 = 4 + 2\log_2 3$.

5. **Answer:** $\mathbf{\frac{3}{10} = .3}$

   Two triangles on a circle intersect if and only if the vertices are interspersed. By this, we mean the ordering $ABEDCF$ is interspersed since the letters ADC don't occur together, but $ABEFDC$ is not because we can rotate the ordering to get an equivalent ordering $DCABEF$, in which the letters of ADC and BEF both occur together. To avoid over-counting always start viewing the possible distributions at point A and move clockwise. There are clearly 5! total ways to order the points. Note that each has equal probability of being chosen. The 6 orientations that are not interspersed are $ACD(BEF)$, $ADC(BEF)$, $AC(BEF)D$, $AD(BEF)C$, $A(BEF)CD$, and $A(BEF)DC$. Note that in each orientation, the triangle BEF can occur in any of 6 possible orderings as well. This gives us 36 total orientations where the vertices do not intersect and thus the probability is $\frac{36}{5!} = \frac{3}{10}$.

6. **Answer: 14094**

   We can count the number of ways to get to any point recursively. You must start at the starting dot and from each point you can come from above it, left of it, above and left diagonally or below and left diagonally provided that these directions are possible. Using this procedure we get the answer of 14094 ways.



7. **Answer: 13**

   Notice that $P(S)$ must be of the form $P(S) = \frac{N(S)}{13!}$ with $N(S)$ an integer, since the number of ways to arrange 13 cards in a row is 13!. So for $P(S) = \frac{1}{n}$, n must satisfy $n \cdot N(S) = 13!$, that is, n divides 13!. The numbers from 1 to 50 that do not satisfy this are 17, 19, 23, 29, 31, 34, 37, 38, 41, 43, 46, 47, and 49, so thirteen values are impossible.

   We now show that the other values for n are possible. Since n divides 13!, let $N(S) = \frac{13!}{n}$. Define S to be a listing of $N(S)$ possible orderings of the thirteen cards (where S is true if one of them is the correct ordering). Each ordering can occur in only one way, so $P(S) = \frac{N(S)}{13!} = \frac{1}{n}$ as desired. Thus, only the thirteen values listed above are impossible to attain.

8. **Answer: 4**

   Euler receives 9 different answers to his question. However, each person shakes hands with at most 8 people. Thus the answers Euler gets are the numbers 0,1,2,. . .,8. Clearly the person who shook hands with no people must be married to the person who shook everyone else's hands. (The person who shook 8 hands clearly shook hands with everyone except himself and his spouse. If someone shook no hands, then she had better be the aforementioned spouse). Likewise, one can reason that the people who shook 1 person's hands and 7 people's hands must be married. This continues until we find that the person who shook four people's hands must be Euler's wife.

9. **Answer: 1722**

   We need to make a list of all scores that are not possible to form with 9, 10 and 17. First, consider scores that can be made with 9 and 10. We can easily make every score greater than 81. To do so, find the closest multiple of 9 less than the number and then replace as many 9s with 10s as necessary to bring the sum up to the desired number. We can always do that if we start with at least 9 9s. Thus, the only numbers that are possibly not attainable are 1, 2, . . ., 80. Scratch off all multiples of 9 and 10. Plus eliminate every number we can make with just 9s and 10s. This leaves us with 1, 2, 3, 4, 5, 6, 7, 8, 11, 12, 13, 14, 15, 16, 17, 21, 22, 23, 24, 25, 26, 31, 32, 33, 34, 35, 41, 42, 43, 44, 51, 52, 53, 61, 62, 71. Next throw out all multiples of 17 and consider what numbers can be made with just one 17 score, two 17 scores, and three 17 scores (with appropriate numbers of 9s and 10s). The remaining list is 1, 2, 3, 4, 5, 6, 7, 8, 11, 12, 13, 14, 15, 16, 21, 22, 23, 24, 31, 32, 33, 41, 42. Thus 41 and 42 are the largest two unattainable scores, and their product is 1722.

10. **Answer: $\frac{3}{8} = .375$**

    Call the answer s. Then

$$\begin{aligned}
s &= (-2)\sum_{n=1}^{\infty} n\left(-\frac{1}{3}\right)^n \\
&= (-2)\sum_{n=1}^{\infty}\sum_{j=1}^{n}\left(-\frac{1}{3}\right)^n \\
&= (-2)\sum_{j=1}^{\infty}\sum_{n=j}^{\infty}\left(-\frac{1}{3}\right)^n \\
&= (-2)\sum_{j=1}^{\infty}\frac{\left(-\frac{1}{3}\right)^j}{1-\left(-\frac{1}{3}\right)} \\
&= \left(-\frac{3}{2}\right)\sum_{j=1}^{\infty}\left(-\frac{1}{3}\right)^j \\
&= \left(-\frac{3}{2}\right)\frac{\left(-\frac{1}{3}\right)}{1-\left(-\frac{1}{3}\right)} \\
&= \frac{3}{8}
\end{aligned}$$