1. How many squares are there in the xy-plane such that both coordinates of each vertex are integers between 0 and 100 inclusive, and the sides are parallel to the axes?

   Answer: 338350

   Solution: Consider the squares with side length l. Notice that l must be between 1 and 100 inclusive. There are $101 - l$ pairs of x-coordinates that satisfy the requirements, and for each such pair, there are $101 - l$ pairs of y-coordinates satisfying the conditions. This gives a total number of $\sum_{l=1}^{100}(101-l)^2 = \sum_{j=1}^{100} j^2 = \frac{100\cdot 101\cdot 201}{6} = \boxed{338350}$.

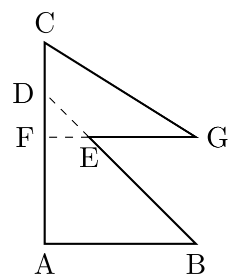
2. According to the Constitution of the Kingdom of Nepal, the shape of the flag is constructed as follows:

   Draw a line AB of the required length from left to right. From A draw a line AC perpendicular to AB making AC equal to AB plus one third AB. From AC mark off D making line AD equal to line AB. Join BD. From BD mark off E making BE equal to AB. Touching E draw a line FG, starting from the point F on line AC, parallel to AB to the right hand-side. Mark off FG equal to AB. Join CG.

   If the length of AB is 1 unit, what is the area of the flag?

   Answer: $\mathbf{\frac{5+3\sqrt{2}}{12}}$

   Solution: Here is a diagram of the construction:



   Because $AB = 1$, $AC = \frac{4}{3}$, $AD = 1$, $BE = 1$, and $FG = 1$. By Pythagoras, $BD = \sqrt{AB^2 + AD^2} = \sqrt{2}$. By similar triangles on DEF and ABD, we have that $\frac{AF}{AD} = \frac{BE}{BD}$ so $AF = \frac{AD\cdot BE}{BD} = \frac{1}{\sqrt{2}}$. Therefore $CF = \frac{4}{3} - \frac{1}{\sqrt{2}}$ and $DF = 1 - \frac{1}{\sqrt{2}}$. Again by similar triangles, $\frac{EF}{AB} = \frac{DF}{AD}$ so $EF = \frac{AB\cdot DF}{AD} = 1 - \frac{1}{\sqrt{2}}$.

   Now, the area of the flag is the sum of the areas of triangle CFG and trapezoid $ABEF$. The area of CFG is $\frac{1}{2}\cdot CF\cdot FG = \frac{2}{3} - \frac{1}{2\sqrt{2}}$. The area of $ABEF$ is $\frac{1}{2}\cdot AF\cdot(AB+EF) = \frac{1}{\sqrt{2}} - \frac{1}{4}$.

   Thus, the area of the flag is $\frac{2}{3} - \frac{1}{2\sqrt{2}} + \frac{1}{\sqrt{2}} - \frac{1}{4} = \boxed{\frac{5+3\sqrt{2}}{12}}$.

3. You have 17 apples and 7 friends, and you want to distribute apples to your friends. The only requirement is that Steven, one of your friends, does not receive more than half of the apples. Given that apples are indistinguishable and friends are distinguishable, compute the number of ways the apples can be distributed.

   Answer: 97944.

   Solution: Note that without the restriction that Steven cannot receive more than half of the apples, there are $\binom{23}{6}$ ways to distribute apples. There are $\binom{23-9}{6} = \binom{14}{6}$ ways to distribute apples given that Steven does receive more than half of the apples. Therefore, there are $\binom{23}{6} - \binom{14}{6} = \boxed{97944}$ ways to distribute apples such that Steven does not receive more than half of the apples.

4. At $t = 0$ Tal starts walking in a line from the origin. He continues to walk forever at a rate of $1\frac{m}{s}$ and lives happily ever after. But Michael (who is Tal's biggest fan) can't bear to say goodbye. At $t = 10s$ he starts running on Tal's path at a rate of n such that $n > 1\frac{m}{s}$. Michael runs to Tal, gives him a high-five, runs back to the origin, and repeats the process forever. Assuming that the high-fives occur at time $t_0, t_1, t_2 ...$, compute the limiting value of $\frac{t_z}{t_{z-1}}$ as $z \to \infty$.

Answer: $\boldsymbol{\frac{n+1}{n-1}}$

Solution: Suppose the two meet at $t_j = x$ a distance x from the origin. Michael would need to run to the origin and back to the same spot, which would take him $2x/n$ time, by which point Tal would move further away by this amount of distance. To reach Tal's new location would require Michael $2x/n^2$ time, and the process repeats until they meet. Thus, it takes Michael a total of $\frac{2x}{n} + \frac{2x}{n^2} + ... = \frac{2x}{n-1}$ time to reach Tal again. Hence $t_{j+1} = x + \frac{2x}{n-1} = \frac{n+1}{n-1}x$ and $\frac{t_z}{t_{z-1}} = \boxed{\frac{n+1}{n-1}}$ is constant for all z.

5. In a classroom, there are 47 students in 6 rows and 8 columns. Every student's position is expressed by (i, j). After moving, the position changes to (m, n). Define the change of every student as $(i - m) + (j - n)$. Find the maximum of the sum of changes of all students.

Answer: 12

Solution: Notice that

$$(i - m) + (j - n) = (i + j) - (m + n)$$

Thus:

$$\begin{aligned}\sum [(i - m) + (j - n)] &= \sum [(i + j) - (m + n)] \\ &= \sum (i + j) - \sum (m + n)\end{aligned}$$

This is the sum of all the student's initial coordinates minus the sum of their final coordinates. The maximum is achieved when the first empty position is the lowest, $(1, 1)$ and the final empty position is the greatest, $(6, 8)$. Since all the other positions remain filled, the total change is:

$$(6 + 8) - (1 + 1) = 12$$

6. Consider the following family of line segments on the coordinate plane. We take $(0, \frac{\pi}{2} - a)$ and $(a, 0)$ to be the endpoints of any line segment in the set, for any $0 \le a \le \frac{\pi}{2}$. Let A be the union of all of these line segments. Compute the area of A.

Answer: $\boldsymbol{\frac{\pi^2}{24}}$

Solution: The family of lines may generally be described as $(\frac{\pi}{2} - a)x + ay = a(\frac{\pi}{2} - a)$ for $0 \le a \le \frac{\pi}{2}$ Take arbitrary x_0 between 0 and $\frac{\pi}{2}$. Let us find the value of a for which $y = \frac{a(\frac{\pi}{2}-a)-(\frac{\pi}{2}-a)x}{a}$ is maximized.

Well, $y = \frac{a(\frac{\pi}{2}-a)-(\frac{\pi}{2}-a)x_0}{a} = \frac{\pi}{2} - a - \frac{\pi x_0}{2a} + x_0$ so $\frac{dy}{da} = -1 + \frac{\pi x_0}{2a^2} \Rightarrow a = \sqrt{\frac{\pi x_0}{2}}$. All is good because this value of a is between 0 and $\frac{\pi}{2}$. Now I will construct a function with $f(x)$ is equal to the maximum value of y possible for given x in the union of line segments. We have just calculated our a for any given x, so plug it into the general equation for the family of lines: $y = \frac{\pi}{2} - a - \frac{\pi x_0}{2a} + x_0 = \frac{\pi}{2} - \sqrt{\frac{\pi x}{2}} - \sqrt{\frac{\pi x}{2}} + x = x - 2\sqrt{\frac{\pi x}{2}} + \frac{\pi}{2}$. The area under this function is our desired $A = \int_0^{\frac{\pi}{2}} x - 2\sqrt{\frac{\pi x}{2}} + \frac{\pi}{2} dx = \boxed{\frac{\pi^2}{24}}$

7. Compute the smallest $n > 2015$ such that $6^n + 8^n$ is divisible by 7.

Answer: 2017

Solution: Note that:

$$8^n = (7+1)^n = 7^n + \binom{n}{1}7^{n-1} + \binom{n}{2}7^{n-2} + \cdots + 1$$

and

$$6^n = (7-1)^n = 7^n - \binom{n}{1}7^{n-1} + \binom{n}{2}7^{n-2} - \cdots \pm 1$$

The sign on the last 1 depends on whether n is odd or even and is $+$ for even and $-$ for odd. Hence $7|(8^n + 6^n)$ if n is odd and thus $\boxed{2017}$ is the answer.

8. Find the radius of the largest circle that lies above the x-axis and below the parabola $y = 2 - x^2$.

Answer: $\mathbf{\frac{2\sqrt{2}-1}{2}}$

Solution: First, note that the center of such a circle must lie on the y-axis and the circle must be tangent to the x-axis and the parabola. Now, consider a circle with center $(0, c)$ and radius r. Since the circle is tangent to the x-axis, necessarily $c = r$. The top half of the circle thus has equation $y = r + \sqrt{r^2 - x^2}$. For the circle to be tangent to the parabola, there needs to be exactly 2 solutions to the equation $r + \sqrt{r^2 - x^2} = 2 - x^2$. We compute:

$$\begin{aligned} r + \sqrt{r^2 - x^2} &= 2 - x^2 \\ r^2 - x^2 &= (2 - r - x^2)^2 \\ r^2 - x^2 &= x^4 - 2(2-r)x^2 + (2-r)^2 \\ x^4 + (2r-3)x^2 + (4-4r) &= 0 \end{aligned}$$

This gives a quadratic in x^2 so the determinant must be zero. Thus, $(2r-3)^2 + 16r - 16 = 0$. This simplifies to $4r^2 + 4r - 7 = 0$ and hence $r = \frac{-4 \pm \sqrt{128}}{8} = \frac{-1 \pm 2\sqrt{2}}{2}$. Since $r > 0$, it follows that the maximum radius is $\boxed{\frac{2\sqrt{2}-1}{2}}$.

9. Let C_1 be the circle in the complex plane with radius 1 centered at 0. Let C_2 be the circle in the complex plane with radius 2 centered at $4 - 2i$. Let C_3 be the circle in the complex plane with radius 4 centered at $3 + 8i$.

Let S be the set of points which are of the form $\frac{k_1+k_2+k_3}{3}$ where $k_1 \in C_1, k_2 \in C_2, k_3 \in C_3$. What is the area of S? (Note: a circle or radius r only contains the points at distance r from the center and does not include the points inside the circle)

Answer: $\mathbf{16\pi/3}$

Solution: First note that the average of the centers of C_1, C_2, C_3 is $7/3 + 2i$. Also, any point in C_1 can be written as m_1, where $|m_1| = 1$, any point in C_2 can be written as $m_2 + (4 - 2i)$, where $|m_2| = 2$, and any point in C_3 can be written as $m_3 + (3 + 8i)$, where $|m_3| = 4$.

Then, any point in S can be written as $\frac{m_1+m_2+m_3}{3} + 7/3 + 2i$. We can translate S by $-7/3 - 2i$ to make $S' = \frac{m_1+m_2+m_3}{3}$ which has the same area as S and is radially symmetric.

Since $\max\left|\frac{m_1+m_2+m_3}{3}\right| = \frac{1+2+4}{3} = 7/3$ and

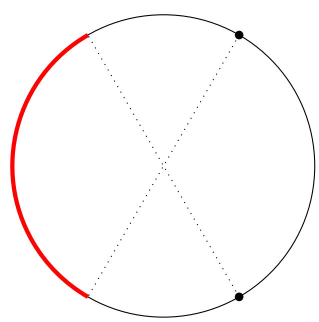
$$\min\left|\frac{m_1+m_2+m_3}{3}\right| = \min\left|\frac{m_3-m_2-m_1}{3}\right| \geq \min\frac{|m_3|-|m_2|-|m_1|}{3} = 1/3,$$

this minimum occurs when $m_1 = -1, m_2 = -2, m_3 = 4$, so $\min\left|\frac{m_1+m_2+m_3}{3}\right| = 1/3$. Thus, $\left|\frac{m_1+m_2+m_3}{3}\right|$ can be any value between $1/3$ and $7/3$ so S' is a disk with radius $7/3$ with a hole of radius $1/3$. Its area is $\pi((7/3)^2 - (1/3)^2) = \pi * 48/9 = \boxed{16\pi/3}$

10. 3 points are independently chosen at random on a circle. What is the probability that they form an acute triangle?

    Answer: $\mathbf{\frac{3}{4}}$

    Solution: It is easier to compute the complement, the probability that the three points form an obtuse triangle. An important observation to make is that 3 points on a circle form an obtuse triangle if and only if they all lie on a semicircle. Now, the position of the first point is irrelevant, so let the first point be fixed. Let θ be the angle between the first and second point. For a fixed θ, the probability that the third point does not lie on any semicircle with the first two points is $\frac{\theta}{2\pi}$ and hence the probability that they *do* form an obtuse triangle is $1 - \frac{\theta}{2\pi}$. The following diagram gives a proof of this: Any point on the black region of the circle lies on a semicircle with



the two points shown. The two extreme cases are when one of the points lies on the diameter of the semicircrle.

Now, the distribution of the angle θ is uniformly distributed between $[0, \pi]$ so it follows that the probability that the three points form an obtuse triangle is

$$\int_0^{\pi} \frac{1}{\pi}\left(1 - \frac{\theta}{2\pi}\right)\, d\theta = 1 - \frac{\pi^2}{4\pi^2} \qquad = \frac{3}{4}$$

and hence the probability that the three points form an acute triangle is $1 - \frac{3}{4} = \boxed{\frac{1}{4}}$.

11. We say that a number is *ascending* if its digits are, from left-to-right, in nondecreasing order. We say that a number is *descending* if the digits are, from left-to-right, in nonincreasing order. Let a_n be the number of n-digit positive integers which are ascending, and b_n be the number of n-digit positive integers which are descending. Compute the ordered pair (x, y) such that $\lim_{n\to\infty} \frac{b_n}{a_n} - xn - y = 0$.

Answer: $\left(\frac{1}{9}, 1\right)$

Solution: We claim that $b_n = \binom{n+9}{9} - 1$ and $a_n = \binom{n+8}{8}$. To see this, we can form an n-digit nonincreasing number by starting with a string $n+9$ blank spaces, "_,_,...,_" and marking 9 of them with an X, denoting X_9 to be the leftmost X, X_8 to be the next leftmost X, with X_1 being the righmost X. Thus our new string is of the form "_,_,..,X_9,_,...,_,X_8,_,.._,X_i,_,...,X_1,_,_,..,". By replacing the blank spaces before X_i but after X_{i+1} with the number i, and replacing the spaces after X_1 with 0, then removing the X's, we produce a sequence of n digits which must be nonincreasing. One can verify this produces a unique nonincreasing number except for the case "X_9,X_8,X_7,X_6,X_5,X_4,X_3,X_2,X_1,_,_,..." which results in 0, which isn't nonincreasing because it's not positive. Similarly one can verify that each nonincreasing number has a unique arrangement of Xs which produces it. It follows that because there are $\binom{n+9}{9}$ ways to arrange the X_i there are $\binom{n+9}{9} - 1$ non-increasing n digit numbers.

The proof for a_n is similar, however we cannot have any zeros in a nondecreasing number, meaning we have only 9 digits to choose from, from which it follows that there are $\binom{n+8}{8}$ nondecreasing numbers with n digits. Thus we have that

$$\frac{b_n}{a_n} = \frac{\binom{n+9}{9} - 1}{\binom{n+8}{8}} = \frac{\frac{(n+9)!}{n!9!} - 1}{\frac{(n+8)!}{n!8!}} = \frac{\frac{(n+9)!}{9} - n!8!}{(n+8)!} = \frac{n+9}{9} - \frac{n!8!}{(n+8)!} = \frac{n}{9} + 1 - \frac{n!8!}{(n+8)!}.$$

Thus

$$\frac{b_n}{a_n} - nx - y = n(\frac{1}{9} - x) + (1-y) - \frac{n!8!}{(n+8)!}.$$

Since $lim_{n\to\infty} \frac{n!8!}{(n+8)!} = 0$, it follows that $\lim_{n\to\infty} \dfrac{b_n}{a_n} - xn - y = 0$ if and only if $x = \frac{1}{9}$, $y = 1$.

12. Let $f(x)$ be a function so that $f(f(x)) = \frac{2x}{1-x^2}$, and $f(x)$ is continuous at all but two points. Compute $f(\sqrt{3})$.

Answer: $\tan \frac{\sqrt{2}\pi}{3}$

Solution: We try to solve a more general problem: if r is a rational number what is the rth iterate of the given function g? Our specific case asks for $r = \frac{1}{2}$ where $g(x) = \frac{2x}{1-x^2}$. To figure out a non-integer iterate, it may really help to find a more general formula for integer iterates first. So what would the nth iterate be? The answer lies in the given function. If you look closely, you must recognize the tangent double angle formula. This makes us suspicious of the tangent function: so let us look at the substitution $x = \tan\theta$. Then $g(\tan\theta) = \tan 2\theta$. So then the nth iteration of $g(\tan\theta)$ must be $\tan(2^n\theta)$. Subsituting back $\theta = \arctan(x)$. we get $g_n(x) = \tan(2^n \arctan x)$, where $g_n(x)$ is the nth iteration. So now we take a leap of faith and hope that plugging in $n = \frac{1}{2}$, a non-integer, gives us the function f that we want. Indeed, if we let $f(x) = \tan(\sqrt{2}\arctan x)$, then $f(f(x)) = \tan(\sqrt{2}\cdot(\sqrt{2}\arctan x)) = \tan(2\arctan x) = \frac{2x}{1-x^2}$.

Plugging in $x = \sqrt{3}$ we get $f(\sqrt{3}) = \boxed{\tan\frac{\sqrt{2}\pi}{3}}$.

13. Compute:

$$\sum_{k=1, k\neq m}^{\infty} \frac{1}{(k+m)(k-m)}.$$

Answer: $\mathbf{\frac{3}{4m^2}}$

Solution: Obviously, we should break this sum in to two parts, a sum up to $m-1$ and one beyond. After writing the summand in partial fractions it is easy to see that the infinite one partially telescopes. We begin with,

$$\begin{aligned}
\sum_{k=1,k\neq m}^{\infty} \frac{1}{(k+m)(k-m)} &= \sum_{k=1}^{m-1} \frac{1}{2m}\left(\frac{1}{k-m} - \frac{1}{k+m}\right) + \sum_{k=m+1}^{\infty} \frac{1}{2m}\left(\frac{1}{k-m} - \frac{1}{k+m}\right) \\
&= \frac{1}{2m}\left[\sum_{n=1}^{m-1}\left(\frac{1}{n-m} - \frac{1}{n+m}\right) + \sum_{n=1}^{\infty}\left(\frac{1}{n} - \frac{1}{n+2m}\right)\right] \\
&= \frac{1}{2m}\left[-\sum_{n=1}^{m-1}\frac{1}{n} - \sum_{n=m+1}^{2m-1}\frac{1}{n} + \sum_{n=1}^{2m}\frac{1}{n}\right] \\
&= \frac{1}{2m}\left[\frac{1}{2m} + \frac{1}{m}\right] \\
&= \frac{3}{4m^2}
\end{aligned}$$

14. Let $\{x\}$ denote the fractional part of x, the unique real $0 \le \{x\} < 1$ such that $x - \{x\}$ becomes integer. For the function $f_{a,b}(x) = \{x+a\} + 2\{x+b\}$, let its range be $[m_{a,b}, M_{a,b})$. Find the minimum of $M_{a,b}$ as a and b ranges along all reals.

Answer: 7/3

Solution: As shifting a and b by $+c$ is equivalent to shifting f by c to left, we can assume $b = 0$. Also $\{x+a\}$ depends only on fractional part of a, so we can assume $0 \le a < 1$.

We can observe that $f_{a,0}(x) = \{x+a\} + 2\{x\}$ has two extrema where $x+a$ is close to 1 and x is close to 1. For the first case $f_{a,0}(1-a-\epsilon) = (1-\epsilon) + 2(1-a-\epsilon) = 3 - 2a - 3\epsilon$, and for the second case $f_{a,0}(1-\epsilon) = (a-\epsilon) + 2(1-\epsilon) = 2 + a - 3\epsilon$. Thus $M_{a,0} = \max(3-2a, 2+a)$, and it has maximum $7/3$ when $a = 1/3$. This can be checked by plotting two equations for $0 \le a < 1$.

15. An ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is tangent to each of the circles $(x-1)^2 + y^2 = 1$ and $(x+1)^2 + y^2 = 1$ at two points. Find the ordered pair (a, b) that minimizes the area of the ellipse.

Answer: $\left(\frac{3\sqrt{2}}{2}, \frac{\sqrt{6}}{2}\right)$

Solution: Since the two circles are symmetric, we can just pick one of them to work on. So we have the system

$$\begin{cases} (x-1)^2 + y^2 = 1 \\ \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \end{cases}$$

Solving the first equation gives us $y^2 = 1 - (x-1)^2$, and clearing denominators gives us $b^2x^2 + a^2y^2 = a^2b^2$ in the second. Substituting for y^2 and simplifying, we obtain

$$(b^2 - a^2)x^2 + 2a^2x - a^2b^2 = 0,$$

which is a quadratic in x.

The tangency condition means that there is a double root, so the discriminant must be zero:

$$(2a^2)^2 - 4(b^2 - a^2)(-a^2b^2) = 0$$

$$a^2b^4 - a^4b^2 + a^4 = 0.$$

Applying the quadratic formula to b^2 gives us

$$b^2 = \frac{a^2 \pm \sqrt{a^4 - 4a^2}}{2}.$$

Since $b < a$ (otherwise there would only be one point of tangency per circle),

$$b^2 = \frac{a^2 - \sqrt{a^4 - 4a^2}}{2}.$$

The quantity we wish to minimize is the area of the ellipse, given by the formula πab. Since a and b are positive, this is minimized if and only if a^2b^2 is minimized. Substituting in b^2 gives us

$$a^2b^2 = \frac{a^4 - a^3\sqrt{a^2 - 4}}{2}.$$

Differentiating this expression produces

$$\frac{4a^3 - \frac{a^4}{\sqrt{a^2-4}} - 3a^2\sqrt{a^2 - 4}}{2},$$

which we set equal to zero. Multiplying by $2\sqrt{a^2 - 4}$, we solve:

$$4a^3\sqrt{a^2 - 4} - a^4 - 3a^2(a^2 - 4) = 0$$

$$a^3\sqrt{a^2 - 4} = a^4 - 3a^2$$

$$a^6(a^2 - 4) = a^8 - 6a^6 + 9a^4$$

$$2a^6 - 9a^4 = 0$$

$$a^4(2a^2 - 9) = 0,$$

so $a^2 = \frac{9}{2}$. Plugging this into the expression for b^2 gives us $b^2 = \frac{3}{2}$. So $(a, b) = \boxed{\left(\frac{3\sqrt{2}}{2}, \frac{\sqrt{6}}{2}\right)}$.