1. Form a triangle ABC with side lengths $AB = 12$, $AC = 8$, and $BC = 15$. Let the altitude from A to BC intersect BC at D and let AE be the angle bisector of $\angle BAC$, where E is on BC. Compute the length of DE.

   Answer: $\mathbf{\dfrac{7}{6}}$

   Solution: From the angle bisector theorem, $\frac{BE}{BA} = \frac{EC}{CA}$. Therefore, $BE = 9$ and $EC = 6$. Let $BD = x$, so $CD = 15-x$ and by the Pythagorean Theorem, we have that $144-x^2 = 64-(15-x)^2$. This has solution $x = \dfrac{61}{6}$, so therefore $DE = \dfrac{61}{6} - 9 = \boxed{\dfrac{7}{6}}$.

2. Consider a unit cube and a plane that slices through it. The plane passes through the mid points of two adjacent edges on the top face, two on the bottom face, and the center of the cube. Compute the area of the cross section.

   Answer: $\mathbf{\frac{3\sqrt{3}}{4}}$

   Solution: Draw the cube to see that the cross-section must be a regular hexagon. To see this you can note that by symmetry the plane must intersect the two vertical edges of the cube at their midpoints. This defines all 6 vertices of the hexagon and it remains to draw them to see that it is regular. One side of the hexagon lies in the top face of the cube between the points on the edges that the plane passes through. The side length is thus $\sqrt{2}/2$. The area of the hexagon is thus $6 \times \frac{(\sqrt{2}/2)^2\sqrt{3}}{4} = \boxed{\dfrac{3\sqrt{3}}{4}}$, 6 times the area of one equilateral triangle of the same side length.

3. Compute the area of the largest square that can be inscribed in a unit cube. You may assume that the square's vertces lie on the edges of the cube.

   Answer: $\mathbf{\dfrac{9}{8}}$

   Solution: It is clear from a sketch that each of the square vertices lie on a different cube edge. By symmetry, if one square vertex lies at a distance x from one cube vertex then each must. That is, each square vertex should divide the cube's edge uppon which it sits into two parts that are the same for each such edge. (In a very fluffy sense this comes from the fact that we need an "even" number of symmetries.) After drawing this, one can see that the side length of the square can be represented in two ways:

   $$\sqrt{2x^2+1} = d = \sqrt{2(x-1)^2}$$

   Thus, $x = 1/4$ so the area is $d^2 = \boxed{\dfrac{9}{8}}$.

4. Inside a circle of radius 1 are three circles of equal radius such that each of them is tangent to the other two and to the large circle. Determine the radius of one of the smaller circles.

   Answer: $\mathbf{2\sqrt{3}-3}$

   Solution: Let O be the center of the larger circle, let P_1, P_2, and P_3 be the centers of the smaller circles, let A_1, A_2, and A_3 be their respective points of tangency to the larger circle, and let B_{12}, B_{23}, and B_{31} be the points of tangency between the corresponding pairs of smaller circles (e.g. B_{12} for the circles with centers P_1 and P_2). Let r be the radius of one of the smaller circles. Then P_1 is on OA_1, OA_1 has length 1, and P_1A_1 has length r. OP_1B_{12} is a 30-60-90 triangle with hypotenuse OP_1, longer leg P_1B_{12}, and shorter leg $B_{12}O$, so because P_1B_{12} has length r, by the properties of 30-60-90 triangles, OP_1 has length $\frac{2}{\sqrt{3}}r$. Thus $1 = r + \frac{2}{\sqrt{3}}r = \frac{2+\sqrt{3}}{\sqrt{3}}r$, so $r = \frac{\sqrt{3}}{2+\sqrt{3}} = \boxed{2\sqrt{3}-3}$.

5. When circles of the same radius are packed into the plane with maximum density they form a regular lattice. Compute the packing density of this arrangement, that is, the fraction of area covered by circles.

   Answer: $\boldsymbol{\frac{\pi}{2\sqrt{3}}}$ **(or equivalently** $\boldsymbol{\frac{\pi\sqrt{3}}{6}}$**)**

   Solution: Let the circles have radius 1. It is easy to see that this is a hexagonal lattice since the angle subtended at the center of one circle by a tangent neighbor is exactly $\frac{\pi}{3}$! Now to compute the density it suffices to calcualte the density inside a hexagon that is centered on a circle and has corners at the centers of its 6 neighbors. This is valid because the plane is tessellated by these hexagons and each one coveres a symmetrical arrangement of circles. In such a hexagon the area covered by circles includes the central circle and 6 arcs of angle $\frac{2\pi}{3}$ of its neighbors. This area is 3π. On the other hand, it is trivial to compute the area of the hexagon since it is made up of 6 equilateral triangles. The distance from the center to a corner of the hexagon is 2 so it has area $6\sqrt{3}$. Therefore, the answer is $\boxed{\dfrac{\pi}{2\sqrt{3}}}$.

6. Consider a unit square $ABCD$. Let E be the midpoint of BC and F the intersection of AC and DE. Compute the area of triangle ADF.

   Answer: $\boldsymbol{\frac{1}{3}}$

   Solution 1: Since AD and BC are parallel, $\angle DAC = \angle ECD$ and $\angle ADE = \angle BCA$ so triangles ADF and CEF are similar. Since E is the midpoint of BC, $\frac{AF}{FC} = \frac{AD}{EC} = \frac{2}{1}$. Thus, the ratio of the area of triangle ADF to the area of triangle CDF is $2:1$ and hence the area of ADF is $\frac{2}{3}$ the area of triangle ACD. The triangle ACD has area $\frac{1}{2}$ since it is half the unit square so triangle ADF has area $\frac{2}{3} \cdot \frac{1}{2} = \boxed{\dfrac{1}{3}}$.

   Solution 2: The line AC has equation $y = -x + 1$ and DE has equation $y = \frac{1}{2}x$. Therefore AC and DE intersect when $-x + 1 = \frac{1}{2}x$ so when $x = \frac{2}{3}$. Thus, triangle ADF has base $AD = 1$ and height $\frac{2}{3}$ and hence area $\boxed{\dfrac{1}{3}}$.

7. Consider a circular sector of unit radius and angle $\arcsin\left(\frac{1}{3}\right)$. Let S be a square inscribed in the sector such that the axis of symmetry of the sector passes through the center of S, is parallel to two of the sides of S, and all four vertices of S are on the boundary of the sector. What is the area of S?

   Answer: $\boldsymbol{\frac{34-20\sqrt{2}}{89}}$

   Solution: Let A denote the center of the sector's corresponding circle. Also, notice that S intersects each of the line segments that make up the boundary of the sector exactly once. Choose one of these intersection points arbitrarily and call it B. Notice that S intersects the arc at two points; denote the one closer to B by C. Observe additionally that the axis of symmetry of the sector intersects the square at two points; denote the one further from A by D and the one nearer by E.

Let $\theta = \arcsin\left(\frac{1}{3}\right)$, and let s denote the length of the side of S so that the desired area is given by s^2.

By construction, $\triangle AEB$ is a right triangle, and $\angle EAB$ has measure $\frac{\theta}{2}$. Observe that $BE = \frac{s}{2}$, and so $AE = \frac{s}{2}\cot\left(\frac{\theta}{2}\right)$. Clearly, $ED = s$, and so $AD = AE + ED = \frac{s}{2}\cot\left(\frac{\theta}{2}\right) + s$. It is also easy to see that $\triangle ADC$ is a right triangle and that $DC = \frac{s}{2}$. Since C lies on the arc, $AC = 1$, and so by the Pythagorean Theorem, $1 = \left(\frac{s}{2}\cot\left(\frac{\theta}{2}\right) + s\right)^2 + \left(\frac{s}{2}\right)^2$, which implies that $s^2 = \left(\left(\frac{1}{2}\cot\left(\frac{\theta}{2}\right) + 1\right)^2 + \frac{1}{4}\right)^{-1}$.

Since $\sin\theta = \frac{1}{3}$, it follows that $\cos\theta = \frac{2\sqrt{2}}{3}$, and thus, by an application of the half-angle formula, $\cot\left(\frac{\theta}{2}\right) = \frac{1+\cos\theta}{\sin\theta} = 3 + 2\sqrt{2}$.

Substituting into the above result gives $s^2 = \left(\left(\frac{1}{2}\left(3 + 2\sqrt{2}\right) + 1\right)^2 + \frac{1}{4}\right)^{-1} = \left(\frac{17}{2} + 5\sqrt{2}\right)^{-1} = \boxed{\dfrac{34 - 20\sqrt{2}}{89}}$.

8. Natasha walks along a closed convex polygonal curve of length 2016. She carries a paintbrush of length 1 and walking all the way around paints all the area as far as she can reach on the outside of the curve. What is that area?

   Answer: $\mathbf{2016 + \pi}$

   Solution: If the curve is convex then she will only be turning one way and by the time she returns to the origin she has turned exactly 2π radians. Therefore she will paint the area of a circle as she turns corners because she paints a sector of the unit circle each time until the sum of the sector angles is 2π. On the straight edges she paints a rectangle of length equal to the length of the segment and width equal to the length of the paintbrush, 1. It is easy to see that the corner sectors and the straight edges do not intersect. So the total area is $2016 \cdot 1 + \pi 1^2$. This is still true if the curve is continuous but we would need calculus to justify it.

9. Four spheres of radius 1 are mutually tangent. What is the radius of the smallest sphere containing them?

   Answer: $\mathbf{1 + \sqrt{\frac{3}{2}}}$

   Solution: Call the containing sphere S and the smaller spheres s_i for $i = 1, 2, 3, 4$. By symmetry of the arrangement S is tangent to each s_i, and the point of tangency must be the point on the sphere s_i farthest away from the center of mass of all 4 small spheres s_i. Said another way, if three of the spheres are placed on a table with the fourth on top of them then the point of tangency on the top sphere is the heighest point from the table. So it sufices to find the distance of this point from the center of mass. Because the spheres are tangent their centers define a tetrahedron of side length 2 by symmetry. The center of mass (CM) of the spheres is also the center of mass of the tetrahedron. We seek the distance from CM to a vertex (a sphere center). Because it is a center of mass, the sum of vectors from CM to each vertex is 0. Therefore one vector's length equals 3 times the projection of the other vectors onto it. Thus, if $\phi > \pi/2$ is the angle between two of the vectors and l is the vector length,

$$l = 3l\cos(\pi - \phi) \implies \cos(\phi) = -1/3$$

   Thus by the law of cosines and the side length of the tetrahedron the vector length is:

$$2^2 = 2l^2 - 2l^2\cos(\phi) \implies l = \sqrt{\frac{3}{2}}$$

So the radius of S is $1+\sqrt{\frac{3}{2}}$.

10. Consider a regular pentagon and connect each vertex to the pair of vertices farthest from it by line segments. The line segments intersect at 5 points to form another smaller pentagon. If the large pentagon has side length 1, compute the area of the smaller pentagon. Express your answer without trigonometric functions.

Answer: $\mathbf{\frac{1}{8}\sqrt{250-110\sqrt{5}}}$

Solution: It is clear by symmetry the the smaller pentagon is also regular. Let the original pentagon have vertices $ABCDE$. Label the smaller pentagon with vertices $A'B'C'D'E'$ where the primed vertex, X', is the single primed vertex farthest away from the unprimed one, X. Call the lengths $A'B' = x$ and $AC' = y$. It is clear from similar triangles that $\frac{AB}{A'B'} = \frac{AC}{B'C}$ or $\frac{1}{x} = \frac{2y+x}{y}$. But it is also clear that $x + y = 1$. This can be seen by the congruent triangles, $\triangle ABC$ and $\triangle AB'C$, because $AD \parallel BC$. Thus, $x = \frac{1}{2}\left(3-\sqrt{5}\right), y = \frac{1}{2}\left(\sqrt{5}-1\right)$.

It is easy to see that the area of a regular pentagon of sidelength d can be calculated by summing the areas of triangles formed with the center and two vertices. The height (apothem) is $d/2\cot(2\pi/10)$. So the area of the whole pentagon is $5 \times \frac{1}{2} \times d/2\cot(2\pi/10) \times d$. So the small pentagon area is:

$$\frac{5}{4}\cot(\pi/5)\left(\frac{1}{2}\left(3-\sqrt{5}\right)\right)^2 = \frac{5}{8}\left(7-3\sqrt{5}\right)\cot(\pi/5)$$

And it remains to calculate $\cot(\pi/5)$. Notice that $\pi/5$ radians is actually $36°$. So we can instead compute $\cot(36°)$. For this is suffices to calculate $\cos(36°)$. Using addition formulas,

$$\begin{aligned} 1 &= \cos(90°) = \cos(72° + 18°) = \cos 72° \cos 18° - \sin 72° \sin 18° \\ &= (2\cos^2 36° - 1)\sqrt{\tfrac{1+\cos 36°}{2}} - 2\sin 36° \cos 36° \sqrt{\tfrac{1-\cos 36°}{2}} \end{aligned}$$

Then $\cos(36°) = u$ satisfies the equation:

$$0 = (2u^2-1)\sqrt{\tfrac{1+u}{2}} - 2\sqrt{1-u^2}\cdot u\sqrt{\tfrac{1-u}{2}}$$

Because $0 < u < 1$ this can be simplified,

$$\begin{aligned} 2\sqrt{1+u}\sqrt{1-u}\cdot u\sqrt{1-u} &= (2u^2-1)\sqrt{1+u} \\ 2u(1-u) &= 2u^2-1 \end{aligned}$$

So $u = \frac{1+\sqrt{5}}{4}$. Thus the cotangent is $\frac{\frac{1+\sqrt{5}}{4}}{\sqrt{1-\left(\frac{1+\sqrt{5}}{4}\right)^2}} = \sqrt{1+\frac{2}{\sqrt{5}}}$. So the area becomes,

$$\frac{5}{8}\left(7-3\sqrt{5}\right)\sqrt{1+\frac{2}{\sqrt{5}}} = \frac{1}{8}\sqrt{250-110\sqrt{5}}$$