1. Find the volume common to 3 cylinders of radius 1 that lie along the three coordinate axes.

   Answer: $\mathbf{8(2-\sqrt{2})}$

   Solution: There are many ways to do this problem. Using multi-variable calculus we proceed as follows. Split the object into 16 congruent pieces. One of these sits in the first octant and is defined by, $\{(r,\theta,z) \mid 0 \le r \le 1,\ 0 \le \theta \le \pi/4,\ 0 \le z \le \sqrt{1-x^2}\}$ where $x = r\cos\theta$. Thus the entire volume is:

$$16\int_0^{\pi/4}\int_0^1\int_0^{\sqrt{1-r^2\cos^2\theta}} r\,dz\,dr\,d\theta = 16\int_0^{\pi/4}\int_0^1 \sqrt{1-r^2\cos^2\theta}\,r\,dr\,d\theta = 16-8\sqrt{2}$$

   Using calculus of only one variable, we want to use the common technique that $V = \int A\,dh$ where A is the area of a cross section and h is the height at which that cross section sits. To do this, divide the region into 8 congruent pieces and consider the 1st quadrant part again. Now we will consider two regions divided by $z > 1/\sqrt{2}$ and $z < 1/\sqrt{2}$. For the first region, slicing the figure with a plane horizontally produces a square cross section. The sides of this square are of length $\sqrt{1-z^2}$. Thus the volume of this region is $\int_{1/\sqrt{2}}^{1}(1-z^2)dz = \frac{1}{12}\left(8-5\sqrt{2}\right)$.

   For the second region we will obtain a cross section that consists of a circular sector and two triangles. This cross section is a subset of the unit disc in the first quadrant. For some point on the circle $\{x, \sqrt{1-x^2}\}$ the region of the cross section is equal to the intersection of the disc and a square of side length x with one corner at the origin and two sides parallel to the coordinate axes. The area of this region is $x\sqrt{1-x^2} + \frac{1}{2}\left(\frac{\pi}{2} - 2\cos^{-1}(x)\right)$. It is also easy to see that $x^2+z^2 = 1 \implies dz = -\frac{x\,dx}{\sqrt{1-x^2}}$. So the volume of this region is,

$$\int_1^{\frac{1}{\sqrt{2}}}\left(\sqrt{1-x^2}\,x + \frac{1}{2}\left(\frac{\pi}{2} - 2\cos^{-1}(x)\right)\right)\frac{-x}{\sqrt{1-x^2}}\,dx = \frac{4}{3} - \frac{7}{6\sqrt{2}}$$

   Summing these two answers gives $2-\sqrt{2}$, finally there are 8 regions, giving us an answer of $\boxed{8(2-\sqrt{2})}$.

2. Let $y(x)$ be the solution to the differential equation $y'' = yx$. The following limit exists:

$$\lim_{x\to\infty}\frac{\ln y}{x^{3/2}}$$

   \- compute it.

   Answer: $\mathbf{\frac{2}{3}}$

   Solution: First of all this is Airy's equation so one can simply recall the asymptotic expansion for $Ai(x)$.

   Because we know that the limit exists call it α. This implies that $y \approx e^{\alpha x^{3/2}}$ for $x \to \infty$. Using the differential equation we have

$$xe^{ax^{3/2}} = \frac{9}{4}a^2xe^{ax^{3/2}} + \frac{3ae^{ax^{3/2}}}{4\sqrt{x}}$$

   Thus $\alpha = \frac{3\sqrt{16x^3+1}-3}{18x^{3/2}}$ and in the limit $x\to\infty$, $\alpha = \boxed{\frac{2}{3}}$.

3. Find f satisfying $4x(1-x)f''+2(1-2x)f'+f=0$ and $f(1)=1,\ f'(1)=1/2$.

Answer: $\boldsymbol{\sqrt{x}}$

Solution: This is Mathieu's equation. Make the substitution $x=\cos^2 y$. Then the equation becomes, $f''(y)+f(y)=0$. The solution to this satisfying the conditions is $\cos(y)$. The rest follows.

Finally suppose that we simply search for the taylor series of f at 1. Differentiating the equation once and evaluating at 1 gives $f''(1)=-1/4$. Then, using the differnetial equation we can write for $n\geq 2$:

$$\begin{aligned}\frac{d^n}{dx^n}(4x(1-x)f''+2(1-2x)f'+f)&=\\ 4x(1-x)f^{(n+2)}+[2(1-2x)+4(1-2x)n]f^{(n+1)}+[1-4n-4n(n-1)]f^{(n)}&=0\end{aligned}$$

So in general

$$f^{(n+1)}(1)=(1/2-n)f^{(n)}(1)$$

The solution to this is the obvious product which can be written as the generalized binomial coefficient: $f^{(n)}(1)=\prod_{m=0}^{n-1}\left(\frac{1}{2}-m\right)=n!\binom{\frac{1}{2}}{n}$. So the series is

$$\sum_{n=0}^{\infty}\binom{\frac{1}{2}}{n}(x-1)^n=\boxed{\sqrt{x}}$$

It should be easy to see how to sum this series from the standard decomposition, $(x+1)^n=\sum_{k=0}^{\infty}\binom{n}{k}x^k$.

4. Compute

$$\int_0^{\frac{\pi}{2}}\frac{dx}{1+\sqrt{\tan x}}.$$

Answer: $\boldsymbol{\frac{\pi}{4}}$

Solution: Split the integral as

$$\int_0^{\frac{\pi}{2}}\frac{dx}{1+\sqrt{\tan x}}=\int_0^{\frac{\pi}{4}}\frac{dx}{1+\sqrt{\tan x}}+\int_{\frac{\pi}{4}}^{\frac{\pi}{2}}\frac{dx}{1+\sqrt{\tan x}}.$$

Apply the change of variables $x\to\frac{\pi}{2}-x$ to the second integral to get

$$\int_{\frac{\pi}{4}}^{\frac{\pi}{2}}\frac{dx}{1+\sqrt{\tan x}}=\int_{\frac{\pi}{4}}^{0}\frac{-dx}{1+\sqrt{\cot x}}=\int_0^{\frac{\pi}{4}}\frac{dx}{1+\sqrt{\cot x}}=\int_0^{\frac{\pi}{4}}\frac{\sqrt{\tan x}dx}{1+\sqrt{\tan x}}.$$

Thus

$$\int_0^{\frac{\pi}{2}}\frac{dx}{1+\sqrt{\tan x}}=\int_0^{\frac{\pi}{4}}\frac{dx}{1+\sqrt{\tan x}}+\int_0^{\frac{\pi}{4}}\frac{\sqrt{\tan x}dx}{1+\sqrt{\tan x}}=\int_0^{\frac{\pi}{4}}\frac{\left(1+\sqrt{\tan x}\right)dx}{1+\sqrt{\tan x}}=\int_0^{\frac{\pi}{4}}dx=\boxed{\frac{\pi}{4}}.$$

5. Compute
$$\int_1^e \frac{\ln(x)}{(1+\ln(x))^2}dx.$$

Answer: $\boldsymbol{\frac{e}{2} - 1}$

Solution: Set $u = 1 + \ln(x)$ then $e^{u-1} = x$ and $e^{u-1}du = dx$ so we have,
$$\int_1^e \frac{\ln(x)}{(1+\ln(x))^2}dx = \int_0^2 \frac{(u-1)e^{u-1}du}{u^2} = \int_0^2 \left(\frac{1}{u} - \frac{1}{u^2}\right)e^{u-1}du$$
Now integrate only the first term by parts and the remaining integrals cancel,
$$\left.\frac{e^{u-1}}{u}\right|_1^2 - \int_0^2 \frac{-1}{u^2}e^{u-1}du - \int_0^2 \frac{1}{u^2}e^{u-1}du = \frac{e}{2} - 1$$

6. Compute
$$\frac{1}{\pi}\int_0^\pi \left(\frac{\sin(10x)}{\sin x}\right)^2 dx.$$

Answer: 10

Solution: Let $z = e^{ix}$ so that $\sin x = \frac{z-z^{-1}}{2i}$ and $\sin(10x) = \frac{z^{10}-z^{-10}}{2i}$. We utilize these representations of $\sin x$ and $\sin(10x)$ to simplify $\frac{\sin(10x)}{\sin x} = \frac{z^{10}-z^{-10}}{z-z^{-1}} = z^{-9}\frac{z^{20}-1}{z^2-1} = z^{-9}\frac{(z^2)^{10}-1}{z^2-1} = z^{-9}(z^{18} + z^{16} + \ldots + 1) = (z^9 + z^7 + \ldots + z + z^{-1} + \ldots + z^{-7} + z^{-9}) = ((z^9 + z^{-9}) + \ldots + (z + z^{-1}))$. The expression in the integral is the square of our "simplification" but before squaring it, I will note several important things that simplify all of the work. First, when we square it, every power of z will be even. Second of all, $z^{2k} + z^{-2k} = 2\cos(2kx)$ has integral 0 over the interval 0 to π. Thirdly, we expand the square $\left(\sum_{k=1}^5 (z^{2k-1} + z^{-2k+1})\right)^2 = \sum_{k=1}^5 (z^{2k-1} + z^{-2k+1})^2 + 2\sum_{1\le k<j\le 5}(z^{2k-1} + z^{-2k+1})(z^{2j-1} + z^{-2j+1}) = \sum_{k=1}^5 (2 + z^{4k-2} + z^{-4k+2}) + 2\sum_{1\le k<j\le 5}\left((z^{2(k+j)-2} + z^{-2(k+j)+2}) + (z^{2(k-j)-2} + z^{-2(k-j)+2})\right)$. So the integral of all of these sums will be 0, with the exception of $\sum_{k=1}^5 2 = 10$. So we have that the expression in our problem is equal to $\frac{1}{\pi}\cdot(10\pi) = 10$, so $\boxed{10}$ is the answer.

A quick comment. Say you have a collection of raffle tickets with n digits in base m (to be clear, if $n = 6, m = 10$ the tickets are $000000, 000001, \ldots, 999998, 999999$). Call a ticket "happy" if n is even and the sum of the first $\frac{n}{2}$ digits is equal to the sum of the last $\frac{n}{2}$ digits. Then the number of happy raffle tickets for any given n and m is equal to $\frac{1}{\pi}\int_0^\pi \left(\frac{\sin(mx)}{\sin x}\right)^n dx$. If one somehow knew this ahead of time, then clearly, for $n = 2, m = 10$ we have $00, 11, \ldots, 99$ are the $\boxed{10}$ happy tickets! Also, it is a fun exercise to prove this formula, and I highly recommend at least trying!

7. Compute
$$\int_0^{1/e} \frac{dx}{\sqrt{-\ln(x) - 1}}.$$

Answer: $\boldsymbol{\frac{\sqrt{\pi}}{e}}$

Solution: Set $u = \sqrt{\ln\left(\frac{1}{ex}\right)}$ then $e^{-u^2-1} = x$ and $-2ue^{-u^2-1}du = dx$ so the integral becomes,

$$\int_0^{1/e} \frac{dx}{\sqrt{-\ln(x) - 1}} = \int_0^{1/e} \frac{dx}{\sqrt{\ln\left(\frac{1}{ex}\right)}} = \int_\infty^0 \frac{-2ue^{-u^2-1}du}{u} = \int_{-\infty}^{\infty} e^{-u^2-1}du = \frac{\sqrt{\pi}}{e}$$

8. We have a triangle ABC with $AB = 2$, $BC = 3$, and $AC = 4$. Consider all lines XY such that X lies on AC, Y lies on BC, and triangle XYC has area half of that of ABC. What is the minimum possible length of XY?

   Answer: $\boldsymbol{\frac{\sqrt{6}}{2}}$

   Solution: Let $x = CX$ and $y = CY$. Then since the area of XYC is half of that of ABC, we must have $xy = \frac{1}{2}AC \cdot BC = 6$. Now, let $\alpha = \angle ACB$. Then by the law of cosines, $XY = \sqrt{x^2 + y^2 - 2xy\cos(\alpha)} = \sqrt{x^2 + y^2 - 12\cos(\alpha)}$. Since $\cos(\alpha)$ is a constant, minimizing XY is equivalent to minimizing $x^2 + y^2$.

   Now, since $xy = 6$, we have $y = \frac{6}{x}$ and hence $x^2 + y^2 = x^2 + \frac{36}{x^2} = \frac{x^4+36}{x^2}$. The minimum is achieved when its derivative is zero. The numerator of the derivative is $4x^2 - 2(x^4 + 36)$. Setting the derivative to equal 0, we see that $x^4 = 36$ and hence $x = 36^{1/4} = \sqrt{6}$ which lies in the range $[2, 4]$ and hence is a valid value of x. Therefore, $x = \sqrt{6}$ and $y = \frac{6}{x} = \sqrt{6}$.

   We now proceed to compute $\cos(\alpha)$ so we may calculate the length of XY when $x = y = \sqrt{6}$. By the law of cosines an triangle ABC, $\cos(\alpha) = \frac{AC^2+BC^2-AB^2}{2AC \cdot BC} = \frac{9+16-4}{24} = \frac{7}{8}$. Therefore, the minimum value of XY is

$$XY = \sqrt{x^2 + y^2 - 12\cos(\alpha)} = \sqrt{6 + 6 - 12 \cdot \frac{7}{8}} = \boxed{\frac{\sqrt{6}}{2}}$$

9. u is a twice differentiable real-valued function on $[-1, 1]$ with $u^2 + 2u'^2 + 2uu'' = 0$, $u(0) = \sqrt{5}$, and $u'(0) = \frac{3}{\sqrt{5}}$. Determine $u\left(\frac{\pi}{4}\right)$.

   Answer: $\boldsymbol{2\sqrt[4]{2}}$

   Solution: Note that for any twice differentiable function u, $(u^2)'' = (2uu')' = 2u'^2 + 2uu''$. Thus the given differential equation is equivalent to $(u^2)'' = -u^2$. The solutions to this equation are all expressions of the form $a\cos t + b\sin t$, where a and b are arbitrary real constants. Thus $u(t)^2 = a\cos t + b\sin t$ for some real constants a and b. Because $u(0) > 0$ and u is continuous on $[-1, 1]$, we can take the nonnegative branch of the square root, giving $u(t) = \sqrt{a\cos t + b\sin t}$. $u(0) = \sqrt{a} = \sqrt{5}$, so $a = 5$. $u'(t) = \frac{-a\sin t + b\cos t}{2\sqrt{a\cos t + b\sin t}}$, so $u'(0) = \frac{b}{a} = \frac{3}{\sqrt{5}}$, and hence $b = 3$. Thus $u(t) = \sqrt{5\cos t + 3\sin t}$, so $u\left(\frac{\pi}{4}\right) = \sqrt{5 \cdot \frac{1}{\sqrt{2}} + 3 \cdot \frac{1}{\sqrt{2}}} = \sqrt{\frac{8}{\sqrt{2}}} = \sqrt{4\sqrt{2}} = \boxed{2\sqrt[4]{2}}$.

10. In tennis, players have two chances to hit a serve in. If the first serve is in, the point is played to completion (until either player wins the point). If the first serve is out, the player hits a second serve. If the second serve is in, the point is played to completion; otherwise, the server automatically loses the point. Andy can precisely control the velocity v of his serve up to 100mph. The faster his serve, the higher the probability of him winning the point if the serve goes in, but the higher the probability that the serve goes out. For a given v, the probability that Andy's serve is in is $p(v) = \frac{150-v}{150}$, and the probability that he wins the point after his

serve goes in is $q(v) = \frac{v}{100}$. Assuming that he chooses optimal velocities for his first and second serves, compute the probability that Andy wins the point.

Answer: $\mathbf{\frac{75}{128}}$

Solution: We have to find the optimal second-serve velocity first. The probability of winning the point after missing the first serve, as a function of v the speed of the second serve, will be

$$p(v) \cdot q(v) = \frac{v(150 - v)}{100 \cdot 150}$$

The derivative of this is $\frac{150-2v}{15000}$, which has only one zero at $v = 75$, so this is the maximum. At this v, Andy's probability of winning the point is $\frac{75}{150} \cdot \frac{75}{100} = \frac{3}{8}$.

Now, given this, the probability of winning the first point as a function of v, the speed of the first serve, will be

$$p(v) \cdot q(v) + (1 - p(v)) \cdot \frac{3}{8} = \frac{v(150 - v)}{100 \cdot 150} + \frac{3v}{8 \cdot 150}$$

The derivative of this is $\frac{150-2v}{100 \cdot 150} + \frac{3}{8 \cdot 150}$, setting this equal to 0 and solving, we find $\frac{150-2v}{100} = \frac{-3}{8} \Rightarrow 8(150 - 2v) = -3 \cdot 100 \Rightarrow 1500 = 16v \Rightarrow v = \frac{375}{4}$.

Finally, this means the probability Andy wins the point is

$$p(v) \cdot q(v) + (1 - p(v)) \cdot \frac{3}{8}\bigg|_{v=\frac{375}{4}} = \frac{3}{8} \cdot \frac{15}{16} + \frac{5}{8} \cdot \frac{3}{8} = \boxed{\frac{75}{128}}$$

Note that this means Andy will always attempt his first serve at 93.75mph, and his second serve at a more conservative 75mph. In fact, real tennis players also hit their second serves significantly slower than their first serves.